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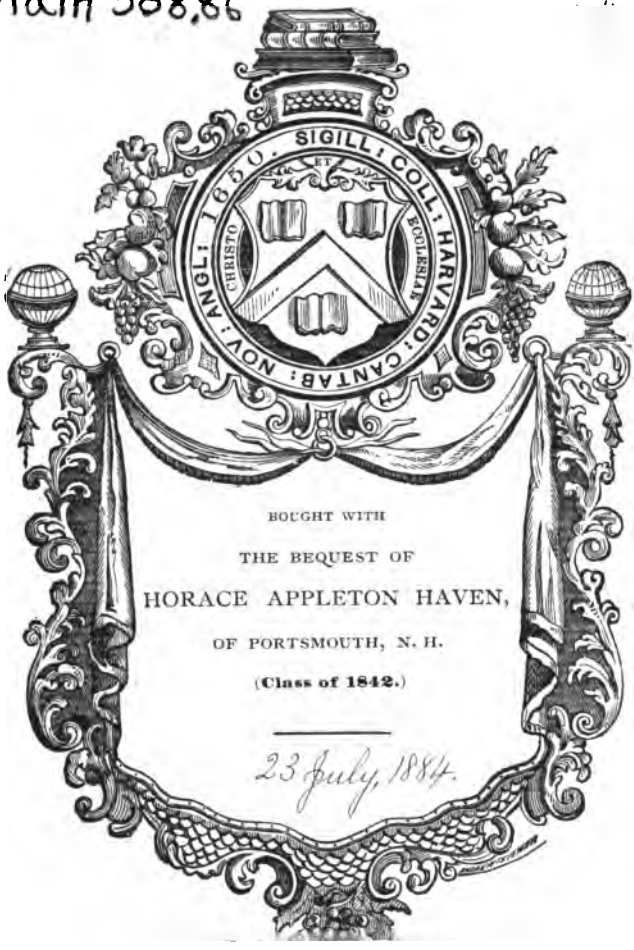

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WITH THEIR

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4019. (E. B. Seitz.)—The points D, E, F are taken at random in the sides BC, CA, AB of the triangle ABC respectively; show that if Δ be the area of the given triangle, and Δ_1 the average area of the triangle formed by joining the centres of the circles AEF, BFD, CDE; then

$$\Delta_1 = \tfrac{1}{24}(7 + \cot^2 A + \cot^2 B + \cot^2 C)\,\Delta. \quad \ldots\ldots\ldots\ldots \quad 24$$

4070. (Professor Clerk-Maxwell.)—A, B, C, D are any four points in a plane; perpendiculars are drawn from D on BC, CA, AB, and a circle is drawn through the feet of these perpendiculars; and, with centre D and radius equal to the diameter of this circle, another circle is drawn: if circles be similarly drawn with their centres at A, B, C, prove (1) that the four circles will intersect by threes in four points a, b, c, d. If the lines joining ABCD be a frame whose weight may be neglected, prove (2) that the stress along any line BC is measured by the corresponding line ad of the second figure. 40

4098. (The Editor.)—A straight line of constant length c moves with with its ends on the curve of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, show (1) that the locus of a point which divides the chord into the two parts λc, μc, which are in the constant ratio $\lambda : \mu$, is

$$a^2y\,(\lambda\mu z^2 - \lambda^2\mu^2 b^2c^2)^{\frac{1}{2}} + b^2x\,(\lambda^2\mu^2 a^2c^2 - \lambda\mu z^2)^{\frac{1}{2}} = \tfrac{1}{2}\,(\lambda - \mu)\,aez^2,$$

where $z^2 = a^2b^2 - a^2y^2 - b^2x^2$; (2) that when the chords are bisected, the locus becomes $\frac{c^2}{4}\left(\frac{a^2y^2 + b^2x^2}{a^4y^2 + b^4x^2}\right) + \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; (3) that when the ellipse becomes a circle, the locus is $x^2 + y^2 = a^2 - \lambda\mu c^2$; and (4) that the area of the locus is always $\pi\,(ab - \lambda\mu c^2)$. 65

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4176. (J. C. Malet, M.A.)—Using the ordinary notation of "Elliptic Functions," prove the following formula for an Elliptic Function of the first kind with an imaginary modulus $k + ik'$, where k is less than unity, $i = \sqrt{(-1)}$, and $k' = (1 - k^2)^{\frac{1}{2}}$, viz.

$$\mathrm{F}\left\{k + ik', \phi\right\} = \mathrm{AF}\,(\lambda, \psi) + \mathrm{BF}\,(\mu, \theta);$$

A, B, λ, μ being real quantities, λ and μ less than unity; find their values and also the values of the amplitudes ψ and θ..... 52

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4192. (T. T. Wilkinson, F.R.A.S.)—Draw a circle to touch two straight lines given in position, and such that, if a tangent be drawn to it parallel to a third line given in position terminating in the lines, the rectangle of the segments made at the point of contact shall be equal to a given space. 76

4196. (Professor Townsend, F.R.S.)—If a conic have triple contact with a tricuspidal quartic, show that—

(*a*) The three quasi-normals at the three points of contact, with respect to the extremities of the bitangent chord, are concurrent.

(*b*) The three second tangents, which, with the three at the points of contact, divide the bitangent chord harmonically, are also concurrent.

(*c*) The line connecting the two points of concurrence passes through the intersection of the three cuspidal tangents, and is divided there and by the bitangent chord in the constant anharmonic ratio of $-3 : 1$. 26

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$$\frac{W}{w} = \frac{b \sin \beta + a \cos(\beta+\gamma) \sin \gamma}{b \cos(\delta+\gamma) + 2a \sin \delta \sin \gamma}. \quad \text{..... } 57$$

4257. (A. Martin.)—A hole $2a$ inches square, is made through the centre of a sphere of radius r; find (1) the surface, (2) the volume removed. 55

4269. (Rev. Dr. Booth, F.R.S.)—If a, b, c, d, e, f, g be chords drawn from any point on the circumference of a circle to the seven angles of an inscribed regular heptagon; prove that

$$(a+g)(b+f)(c+e) = d^3, \quad a+c+e+g = b+d+f \text{.....(1, 2).} \quad 53$$

4275. (W. H. H. Hudson, M.A.)—If S, H be the poles of a Lemniscate, P any point on it, C the middle point of SH, and if PH′, PS′ be taken on PS, PH, equal respectively to PH, PS; prove that the normal at P bisects S′H′. 46

4283. (T. T. Wilkinson, F.R.A.S.) — ABCDE &c. is an irregular unclosed polygon, and Y, Z are two given points within it. It is required to draw YM, MN, NO, &c., meeting the sides AB, BC, CD, &c., at M, N, O, &c., and making the angle YMA = BMN, the angle MNB = CNO, &c., so that the last line shall pass through the given point Z. 99

4292. (The Editor.)—From the ends of the base of a triangle straight lines are drawn—in the same or in a different direction—

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parallel to the opposite sides, and proportional in length to the *adjacent* sides; show (1) that the straight lines joining the ends of these parallels with the remote ends of the base, intersect each other on one of two straight lines which pass through the vertex of the triangle, and divide the base internally and externally in the duplicate ratio of the adjacent sides. Also show (2) that if the parallels are proportional in length to the *opposite* sides, the locus of the intersections will be a line from the vertex bisecting the base, or else parallel to the base. 18

4293. (Rev. Dr. Booth, F.R.S.)—At the ends B, C of the base of a triangle ABC, perpendiculars BD, CE are drawn to the sides AB, AC, having a constant ratio to these sides; prove that the lines which join the ends of these perpendiculars with the opposite angles meet on the perpendicular from A on the base BC. 39

4294. (Professor Wolstenholme, M.A.)—Prove that

$$2^n(x-1)^n-(n-1)\,2^{n-2}(x-1)^{n-2}(x+1)^2$$
$$+\frac{(n-2)(n-3)}{\lfloor 2}(x-1)^{n-4}(x+1)^4-\ldots\ldots \text{to } \frac{n+1}{2} \text{ or } \frac{n}{2}+1 \text{ terms},$$
$$\equiv (n+1)(x^n)-\frac{2(n+1)\,2n}{\lfloor 3}x^{n-1}+\frac{2(n+1)\cdot 2n\cdot 2(n-1)\cdot 2(n-2)}{\lfloor 5}x^{n-2}\cdot$$
$$-\ldots\ldots\ldots \text{ to } (n+1) \text{ terms},$$

n being any integer. 30

4297. (J. J. Walker, M.A.)—If h_1, h_2, h_3, h_4, h_0 are the distances of the four corners and the intersection of diagonals respectively of a plane quadrilateral from any other plane, prove that the distance of the centre of gravity of its area is equal to

$$\tfrac{1}{2}(h_1+h_2+h_3+h_4-h_0). \ldots\ldots 55$$

4301. (E. B. Elliott.)—Two points are taken on an ellipse, all values of their eccentric angles being equally likely, and the tangents at them are drawn. Find the chance that these intersect in the area contained between two given ellipses, similar, similarly situated, and concentric with the first. 49

4316. (Dr. Hart.)—Find n numbers such that, if each number be subtracted from the cube of their sum, the results shall be cubes..... 71

4331. (A. Martin.)—Find rational triangles whose sides, the radius of its inscribed circle, and the radii of "Malfatti's circles," shall all be rational numbers. 70

4350. (C. W. Merrifield, F.R.S.)—Of all convex polyhedra having their dihedral lengths and their edges all equal, prove that there are, besides the four Platonic or regular solids, only two which can be inscribed, and two which can be circumscribed, to a sphere. 27

4352. (H. S. Monck.)—If δ be the deviation of a ray of light transmitted at the polarising angle of any medium, prove that the greatest difference of phase between the two components of a ray of polarised light which can be produced by total reflexion at the surface of the same medium is 2δ. 25

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4354. (R. Tucker, M.A.)—Solve the equations

$$-x^2+xy+xz=a=4 \quad \ldots\ldots (1),$$
$$-y^2+xy+yz=b=-20 \quad \ldots\ldots (2),$$
$$-z^2+xz+yz=c=-8 \quad \ldots\ldots (3).$$ 19

4355. (H. H. Hudson, M.A.)—A curve is such that the radius vector makes half the angle with the normal that it does with a fixed line; find the chord of curvature through the pole. 21

4359. (N'Importe.)—Prove the truth of the following construction for inscribing a square in a given triangle ABC: Draw BD perpendicular to AC; complete the rectangle BDCE, and then the square CEFG; then a line parallel to AC through the intersection (O) of AF and CE will give one side of the required square. 37

4360. (A. Martin.)—From the ends of a diameter of a given circle lines are drawn to a point in the circumference, and on these as diameters circles are drawn; prove that the mean area common to these two circles, in parts of the area of the given circle, is $\frac{1}{4}-\frac{2}{\pi^2}$. 47

4368. (Rev. Dr. Booth, F.R.S.)—A spherical triangle is inscribed in one circle, and circumscribed to another circle of a sphere whose spherical radii are r and R, and Δ is the arc of a great circle joining their centres; show (1) that

$$\sin^2\Delta=\cos^2 R\cos^2 r(\tan^2 R-2\tan R\tan r);$$

and (2) that when the sphere becomes a plane, then

$$\Delta^2=R^2-2Rr. \ldots\ldots 64$$

4376. (W. Niven, M.A.)—A uniform circular wire of radius a, charged with electricity of line-density e, surrounds an uninsulated concentric spherical conductor of radius c; prove that the electrical density at any point of the surface of the conductor is

$$-\frac{e}{2c}\left(1-5\cdot\frac{1}{2}\cdot\frac{c^3}{a^3}Q_2+9\cdot\frac{1\cdot3}{2\cdot4}\frac{c^4}{a^4}Q_4-13\cdot\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{c^6}{a^6}Q^6+\ldots\ldots\right);$$

where Q_1, Q_2, Q_3, ... denote the zonal harmonics at the point, the pole of the plane of the wire being the pole of the harmonics. 44

4378. (W. H. H. Hudson, M.A.)—If A, A′ are different powers of the prime factor a, and B, B′ of another prime factor b, and if AB, AB′, A′B, A′B′, have n, m, p, q divisors, prove that abABA′B′ has $(n+m+p+q)$ divisors. 36

4380. (M. Collins, LL.D.)—Prove that the equation

$$x^2+D^m=(N^2+D)y^2$$

is always possible in rational numbers for x and y when N and D are rational, and m is an odd integer, and that x and y can be found in integers when N and D are integers. 23

4384. (J. L. McKenzie.)—AA′ is a diameter of a conic, and Q any point in the tangent at A. The line A′Q cuts the conic in B; prove that the tangent at B bisects AQ. 22

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4386. (J. J. Sylvester, F.R.S.)—Given a cubic equation in x and a quadratic equation in y, express under the form of a determinant the left-hand side of the equation, whose roots are $lx+my$..... 31

4394. (T. T. Wilkinson, F.R.A.S.)—When two points are given in the circumference of a given circle, prove that another circle may be found, given in position, such that, if any two lines be drawn from the given points to intersect in its circumference and meet the given circle in two points, the line joining these points will be of constant length. 32

4396. (J. J. Walker, M.A.)—Find at what points on a conic the angle between the tangent and the line drawn to a fixed point on the conic is a maximum or minimum. 29

4397. (S. Tebay, B.A.)—A circle is drawn at random within a given circle; show that (1) the probability that it contains the centroid of the remaining figure is $\frac{5}{108}$; and (2) the analogous probability for a sphere is $\frac{5}{1250}$. 27

4399. (J. Griffiths, M.A.)—Let A_1, B_1, C_1 be the vertices of a triangle inscribed in another given triangle ABC so as to be homologous with it; P any point on their axis of homology; X, Y, Z the points where the lines AP, BP, CP meet the sides BC, CA, AB. Show that the envelope of the axis of homology of the triangles ABC, XYZ is the conic which touches the sides of the former at the points A_1, B_1, C_1..... 39

4401. (A. Martin.)—Find rational triangles whose sides and the lines drawn from the angles to the centre of the inscribed circle shall all be whole numbers. 102

4402. (W. H. H. Hudson, M.A.)—If r, p, ρ be radius vector, perpendicular on the tangent, and radius of curvature of a curve, and r', p', ρ' be corresponding quantities for an inverse curve; prove that $\frac{r}{\rho}+\frac{r'}{\rho'}=\frac{p}{r}+\frac{p'}{r'}$. 107

4405. (T. Cotterill, M.A.)—In a spherical triangle, if h be the perpendicular from the angle C on the side c, which is bisected internally in F and externally in F', prove that

$\sin^2\frac{1}{2}c\cos^2 h+\cos a\cos b=\cos^2 FC$, and $\cos^2\frac{1}{2}c\cos^2 h-\cos a\cos b=\cos^2 F'C$.

Hence, find the numerical limits of the expressions

$\sin^2\frac{1}{2}c+\cos a\cos b$, and $\cos^2\frac{1}{2}c-\cos a\cos b$..... 96

4406. (J. L. McKenzie.)—Find the least triangles whose sides are integers in arithmetical progression, and the perpendiculars on the sides from the opposite angles integers in harmonical progression. 54

4414. (Rev. Dr. Booth, F.R.S.)—A triangle circumscribes a parabola; prove that the area of the triangle whose vertices are the three points of contact, is double that of the circumscribing triangle. 17

4420. (J. L. McKenzie.)—Prove that, in a system of confocal ellipses, the envelope of a normal that makes with the major axis an angle whose sine is bc^{-1}, is a four-cusped hypocycloid, with two opposite cusps at the foci of the system..... 96

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4421. (B. Williamson, M.A.)—If any straight line drawn through the pole of a spiral of Archimedes, meet two consecutive branches of the curve in the points A, B, and A′, B′ respectively; prove that the area between the line and the two branches is half that of the ellipse whose semi-axes are AA′ and BA′..... 29

4422. (W. B. Davis, B.A.)—Prove (1) that the sum of d odd squares is of the form $8b + d$; and (2) ascertain whether the converse is true. 25

4423. (T. T. Wilkinson, F.R.A.S.)—The diameter of a semicircle is divided into two unequal parts, on which two other semicircles are described; find a circle whose area shall be equal to the curvilinear area contained between the three semicircles. 67

4424. (J. J. Walker, M.A.)—Forces P, Q act on a point A; and B, C, D are the points in which any transversal meets the lines of action of these forces and their resultant respectively;

prove that $\frac{R}{AD} = \frac{P}{AB} + \frac{Q}{AC}$..... 22

4425. (Christine Ladd.)—Find the nature of the *catenaria volvens*, or the figure which a perfectly flexible chain of uniform density and thickness will assume when it revolves with a constant angular velocity about an axis, to which it is fastened at its extremities, in free and non-gravitating spaces. 85

4426. (W. H. H. Hudson, M.A.)—A particle moves in a resisting medium, the law of resistance being expressed by kv^2; find what initial velocity would be reduced one-half in a unit of time. Of what dimensions is k?..... 38

4432. (N'Importe.)—An indefinite number of ellipses are drawn with an endless string, of length $2a$. One focus is fixed and the other moves on a straight line; find the equation to the locus of the intersections of the ellipses. 33

4436. (Sir James Cockle, F.R.S.)—Assuming, as is proved in the *Messenger* for November 1873 (Vol III. pp. 108 *et seq.*), that

$$\frac{d^2u}{dx^2} + 2a\frac{du}{dx} = bu = 0 \quad \ldots\ldots\ldots\ldots\ (1)$$

is soluble when $\frac{d^2b}{dx^2} + 10a\frac{db}{dx} + 4b\frac{da}{dx} + 3b^2 + 24a^2b = 0$ ... (2);

find a modulus c which shall render

$$\frac{d^2u}{dx^2} + \tfrac{1}{2}\left\{\frac{1}{\sqrt{3}} - (\tan \tfrac{1}{2} amx)^2\right\}u = 0 \ldots\ldots\ (3)$$

soluble by co-resolvents, the elliptic integral being of the first species. 41

4443. (J. J. Walker, M.A.)—Tangents are drawn to an ellipse at P, P′, and from the centre O, lines OQ, OQ′ parallel to either tangent, and meeting the other in Q, Q′ respectively; prove that the triangle contained by the semi-axes is a mean proportional between the triangles POP′, QOQ′. 51

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4445. (E. B. Elliott, B.A.)—In the equation $ax^2 + 2hxy + by^2 = 1$, a, h, and b may have any real values, all being equally likely. Show that the chances of the conic represented in rectangular coordinates being an ellipse, an imaginary conic, or an hyperbola, are respectively $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{3}$; and in the case of an hyperbola distinguish between the cases in which the curve meets in real points, both coordinate axes (chance $\frac{6}{\pi^2}$), neither axis ($\frac{6}{\pi^2}$), or one and only one $\frac{1}{2}$)..... 92

4451. (Dr. Hart.)—Find the equation to the curve that will cut at an angle of 45° any number of circles having their centres on a given straight line, and their circumferences passing through a given point in that line..... 105

4456. (Whitworth's *Choice and Chance*.)—If three numbers be named at random, prove that they are just as likely as not to be proportional to the sides of a possible triangle. 50

4458. (Professor Cayley.)—Find (1) the intersections of the two quartic curves

$\lambda(ab-xy)^2 = abx(a-y)(b-y)$, $\mu(ab-xy)^2 = aby(a-x)(b-x)$;

and (2) trace the curves in some particular cases; for instance, when $a=1$, $b=2$, $\lambda=1$, $\mu=-2$. 60

4460. (Professor Crofton, F.R.S.)—Show that a perfectly flexible, heavy, uniform string may be made to rest (in unstable equilibrium) in the form of an arch. Find, also, the form of an arch composed of a number of smooth equal spheres, and the curve formed by the arch when the number of spheres is continually increased. 46

4461. (Professor Wolstenholme, M.A.)—A triangle ABC is circumscribed about a fixed ellipse of focus S, such that the angles SBC, SCA, SAB are all equal: prove that each of them $= \sin^{-1}(\frac{1}{2}a^{-1}b)$, and that the angular points of the triangle lie on one of two fixed circles whose radius is $2a^2b^{-1}$; $2a$, $2b$ being the axes of the ellipse. 58

4462. (Rev. Dr. Booth, F.R.S.)—In Maclaurin's "*Tractatus de Linearum Geometricarum proprietatibus generalibus*," p. 11, it is shown by Algebra that in any curve the sum of the reciprocals of the subtangents made by a vector revolving round a given point is a constant. Give a geometrical proof of this theorem in the case of the conic sections. 48

4463. (The Editor.)—Let a, b be two conjugate semi-diameters of an ellipse; and (x', y') the coordinates, in reference thereto, of a variable point in the curve: show that the envelope of a series of ellipses whose semidiameters are coincident in direction with a, b, and in magnitude are mean proportionals between a, x' and b, y', is given by the projective or tangential equation

$$\left(\frac{x}{a}\right)^{\frac{4}{3}} + \left(\frac{y}{b}\right)^{\frac{4}{3}} = 1, \quad \text{or} \quad a^4\xi^4 + b^4\upsilon^4 = 1. \text{ .....} \quad 93$$

4465. (T. Cotterill, M.A.)—If a circle has its centre on a conic and passes through a focus, prove that the diameters of the circle through its points of intersection with the directrix of the focus are parallel to the asymptotes of the conic. 83

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4466. (R. F. Scott, B.A.)—If at each point of an ellipsoid a distance k^2p^{-1} be measured along the normal, p being the perpendicular from the centre on the tangent plane, prove that the locus of the points thus obtained is another ellipsoid, the envelope of which for different values of k is the centro-surface of the original ellipsoid..... 42

4467. (J. Jameson.)—DEFG is a square inscribed in a triangle ABC whose base BC is given in magnitude and position; BE, CD meet in O; and GE, FD meet in S; show that the straight line OS always passes through the vertex of a semicircle on BC as diameter 72

4472. (J. J. Walker, M.A.)—Show that the line joining any point outside a conic with its centre and the common chord of the two circles drawn through the point, one passing through the points of contact of tangents from it, the other through the foci, are equally inclined to the lines joining the given point with the foci 101

4477. (W. H. H. Hudson, M.A.)—Prove that the expression

$$(xyz + x^2y - y^2z + z^2x)^2 + (xyz + xy^2 + yz^2 - zx^2)^2$$

is unaltered in value by an interchange of the letters x, y, z. ... 56

4479. (Christine Ladd.)—The radii of the fore and hind wheels of a coach are r and R, and a is the distance between their centres. A particle driven from the highest point of the hind wheel falls on the highest point of the fore wheel; find the velocity of the coach. 51

4480. (G. S. Carr.)—A heavy cylinder rotating rapidly on its axis is projected upwards in the direction of its axis, which is inclined to the horizon. Assuming that the resultant pressure of the air upon the cylinder perpendicular to its axis varies as the square of the velocity of the cylinder in that direction, and that the friction against the surface of the cylinder varies as the pressure of the air upon it: show that the distance of the projectile from the vertical plane of projection after the time t will be the same for all initial velocities of projection, and will be

$$c\left\{\tfrac{1}{2}t^2 + \frac{t}{a} - \frac{1}{a^2}\log\ \tfrac{1}{2}(e^{2at}+1)\right\},$$

where c and a are constants.

The axis of the cylinder is supposed to retain its original direction, and the resistance of the air to the lateral motion itself is left out of the calculation. 97

4484. (Rev. Dr. Booth, F.R.S.)—If a parabola is circumscribed by a quadrilateral two of whose sides are fixed, and the other two are variable in position, prove that the latter intercept on the former segments which are always in a constant ratio to each other. 84

4485. (Professor Townsend, F.R.S.)—The top of a verticle rectangular wall, of uniform material and construction, sustains the oblique thrust of an ordinary inclined roof, supposed uniformly distributed over its entire area; show (1) that the "line of pres-

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sure" of the entire force resulting from the propagated thrust and superincumbent weight, throughout any transverse section of the wall, is a rectangular hyperbola, passing through the middle point of the upper side of the section, and having its asymptotes horizontal and vertical; and (2) determine, for a given thrust of roof and weight of material, the minimum thickness which would enable the wall to be raised to any height without being overturned by the roof. 68

4492. (R. F. Scott, B.A.)—Prove that the lines of curvature of the cone $ax^n + by^n + cz^n = 0$ are its generators and the curves of intersection with spheres drawn round the vertex as centre. ... 101

4498. (H. Murphy.)—Given two sides of a triangle, the product of whose base by the square of the perpendicular is a maximum; prove that the product of the tangents of the angles at the base is 2. 86

4499. (M. Collins, LL.D.)—Required a short rule or method for finding the remainder of the division (without the trouble of the long actual work) when a great whole number expressed by (or containing) say a hundred or more arithmetical figures is divided by 73 or 47. 87

4500. (J. J. Walker, M.A.)—Prove that the intercept on the diameter of the circle circumscribing the plane triangle ABC between the angle A and the opposite side BC is equal to

$$\frac{a\cos A + b\cos B + c\cos C}{\sin 2B + \sin 2C}.$$ 112

4501. (Christine Ladd.)—The distance from A to B is a miles. A man at A travels one-mth of the distance to B the first day; the next day, one-nth of the distance back to A; the third day, one-mth of his distance to B; the fourth day, one-nth of the distance back to A, as so on. How far will he be from A at the end of r days, (1) when r is even, (2) when r is odd? 103

4505. (Sir James Cockle, F.R.S.)—Find a complete solution of the partial biordinal

$$q^2 r - A^2 t + \frac{A^2}{\eta}\frac{d\eta}{dy} q = 0 \quad \text{..... (1),}$$

η being a function of y. 88

4507. (Dr. Booth, F.R.S.)—Eliminate x, y, z from

$$D\xi = Ax + B_1 z + B_2 y + C, \quad Dv = A_1 y + B_2 x + Bz + C_1,$$
$$D\zeta = A_2 z + By + B_1 x + C_2, \text{ and } x\xi + yv + z\xi = 1,$$

where $$D = 1 - Cx - C_1 y - C_2 z;$$

or, in other words, given the projective equation of a surface of the second order, find its tangential equation referred to the same axes. 95

4508. (Professor Wolstenholme, M.A.)—A large number of equal particles are fastened at unequal intervals to a fine string. and then collected into a heap at the edge of a smooth horizontal table with the extreme one just hanging over the edge; the

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intervals are such that the times between successive particles being carried over the edge are equal; prove that if c_n be the interval between the nth and $(n+1)$th particle, and v_n the velocity just after the $(n+1)$th particle is carried over, then

$$c_1^{-1}c_n = v_1^{-1}v_n = n. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 81$$

4519. (A. Martin.)—Find the equation to the envelope of equal chords of a given ellipse. 76

4520. (A. B. Evans, M.A.)—Find the least integral values of x and y that will satisfy the equation $x^2 - 953y^2 = -1$. 78

4523. (J. C. W. Ellis, M.A.)—A pavement is formed of equal elliptical slabs of white marble, their major axes being each $2a$ and pointing to the north, and minor axes $2b$. Any four contiguous slabs have their centres at the angles of an oblong whose sides are $2a$ and $2b$. The interstices are filled up with black marble. A black elliptical lamina, whose axes are $2ma$, $2mb$, and a white one, whose axes are $2na$, $2nb$, are dropped at random on the pavement, and take up positions with their major axes pointing to the north. Find the chance of the black one being entirely on a white surface, and the white one on a black. If $a = b$, and the radius of a silver coin be $\frac{1}{3}(2\sqrt{3}-3)\,a$, and of a copper one $\frac{1}{10}a$; show that the chance of the silver lying wholly on the black, and the copper wholly on the white, is $\frac{3}{400}\pi(9-\pi-3\sqrt{3})$. 77

4530. (Professor Wolstenholme, M.A.)—A triangle ABC is inscribed in a circle, and the tangent at A meets BC in Q; prove (1) that a straight line drawn through Q perpendicular to the bisector of the angle A will meet the circle in two points such that their distances from B, A, C are in geometrical progression; also (2) that the straight line through Q parallel to the bisector of the angle A will meet the circle in two points possessing the same property, provided that $a^2 > 4bc$. 108

4532. (Professor Townsend, F.R.S.)—Four material particles a, b, c, d, connected with a common point O by four inextensible cords OA, OB, OC, OD, repel each other with forces varying directly as their masses and mutual distances conjointly; shew that, in their configuration of relative equilibrium,

$$\text{BCDO} : \text{CDAO} : \text{DABO} : \text{ABCO} = a : b : c : d,$$

each tetrad of letters representing the volume of the tetrahedron of which its constituents are the vertices. 90

4554. (Professor Wolstenholme, M.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \log \sin x \,.\, \log \cos x\, dx = \tfrac{1}{4}\pi(\log 2)^2 - \tfrac{1}{96}\pi^3 \quad \ldots\ldots\ldots(1);$$

$$\int_0^{\frac{1}{2}\pi} \log(\cos x + \sin x) \log(\cos x - \sin x)\, dx = \tfrac{1}{16}\pi(\log 2)^2 - \tfrac{1}{96}\pi^3 \quad \ldots(2). \quad 99$$

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MATHEMATICS

FROM

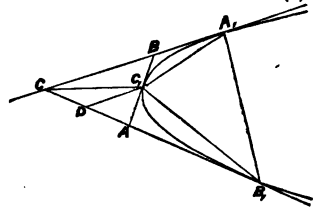
THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

4414. (Proposed by Rev. Dr. Booth, F.R.S.)—A triangle circumscribes a parabola; prove that the area of the triangle whose vertices are the three points of contact, is double that of the circumscribing triangle.

I. *Solution by the* Proposer.

In Booth's *New Geometrical Methods*, pp. 43, 47, it is shown that if a parabola is referred to a pair of tangents as axes of coordinates, its tangential equation is $g\xi v + h\xi + h_1 v = 0 \equiv V$ ..... (a).



Let CA, CB, be the axes of coordinates, tangents to the curve. Let $CB = b$, $CA = a$, and as AB is a tangent to the curve, we may put $a^{-1} = \xi$, $b^{-1} = v$ in (a). Hence the equation of condition is $g + hb + h_1 a = 0$ ..... (b).

The triangle $A_1B_1C_1$ is manifestly the difference between the triangle CA_1B_1 and the triangles CC_1B_1 and CC_1A_1, or the triangle

$$A_1B_1C_1 = CA_1B_1 - CC_1B_1 - CC_1A_1 \text{ ..... } (c).$$

Now the value of the tangent CA_1 is found by putting $\xi^{-1} = 0$, in the general equation (a); hence $CA_1 = -\frac{g}{h}$; and in like manner $CB_1 = -\frac{g}{h_1}$; therefore if $\sin ACB = n$, the area of the triangle A_1CB_1 is $= \frac{ng^2}{hh_1}$.

The projective coordinates CD, C_1D, or the x, y of the point of contact c_1 are found from the equations of transition (see p. 14 of the above named work) $x = \frac{dV}{d\xi} \div \left(\frac{dV}{d\xi}\xi + \frac{dV}{dv}v\right)$, $y = \frac{dV}{dv} \div \left(\frac{dV}{d\xi}\xi + \frac{dV}{dv}v\right)$,

whence $x = \frac{gv + h}{g\xi v}$, $y = \frac{g\xi + h_1}{g\xi v}$;

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or putting $\frac{1}{a}$ for ξ and $\frac{1}{b}$ for v, we shall have $x = a + \frac{hab}{g}$; and as $CA_1 = -\frac{g}{h}$, the area of the triangles CC_1A_1, CC_1B_1 are

$$-n\left(\frac{ag}{h} + ab\right), \quad -n\left(\frac{bg}{h_1} + ab\right);$$

consequently the area of the triangle

$$A_1B_1C_1 = \frac{ng^2}{hh_1} + \frac{nag}{h} + \frac{nbg}{h_1} + 2nab = \frac{ng}{hh_1}(g + hb + h_1a) + 2nab$$
$$= 2nab = 2ABC, \text{ since } g + hb + h_1a = 0.$$

II. *Solution by the* Rev. J. R. Wilson, M.A.

Let L, M, N be the points of contact of the tangents PQ, QR, RP respectively. Then (for shortness writing par. area LN to denote the area included between the curve and the chord LN) we have

par. area LN − par. area LM − par. area MN
$$= \tfrac{2}{3}\triangle PLN - \tfrac{2}{3}\triangle QML - \tfrac{2}{3}\triangle RMN,$$

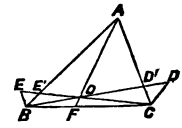
therefore $\triangle LMN = \tfrac{2}{3}\triangle PQR + \tfrac{2}{3}\triangle LMN,$

therefore $\triangle LMN = 2\triangle PQR.$

4292. (Proposed by the Editor.)—From the ends of the base of a triangle straight lines are drawn—in the same or in a different direction—parallel to the opposite sides, and proportional in length to the *adjacent* sides; show (1) that the straight lines joining the ends of these parallels with the remote ends of the base, intersect each other on one of two straight lines which pass through the vertex of the triangle, and divide the base internally and externally in the duplicate ratio of the adjacent sides. Also show (2) that if the parallels are proportional in length to the *opposite* sides, the locus of the intersections will be a line from the vertex bisecting the base, or else parallel to the base.

I. *Solution by* S. Forde; Belle Easton; *and others.*

Let CD, BE be the lines drawn parallel to AB, AC respectively; and through O the intersection of BD and CE draw AO cutting BC in F. Then, since ABD′, D′CD are similar triangles, we have



$$AD' : D'C = AB : DC,$$

and similarly $AE' : E'B = AC : BE;$

therefore $$\pm\frac{BF}{CF} = \frac{E'B \cdot D'A}{AE' \cdot CD'} = \frac{BE}{AC} \cdot \frac{AB}{DC} = \frac{AB^2}{AC^2} \text{ or } 1,$$

according as the parallels are proportional to *adjacent* or to *opposite* sides; which proves the theorems in the Question.

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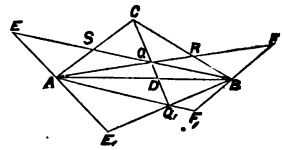
II. *Solution by the* PROPOSER.



1. Let ABC be the triangle, and AE, BF parallel to BC, AC, and proportional in length to these sides, so that $AE : AC = BF : BC = \lambda : \mu$, then the trilinear equation of AF will be

$$\pm\frac{\beta}{\gamma} = \frac{\sin FAC}{\sin FAB} = \frac{\sin AFB}{\sin FAB} = \frac{\mu c}{\lambda a},$$

or $\lambda a\beta \mp \mu c\gamma = 0$ ..... (1).

Similarly, the equation of BE is $\lambda b\alpha \pm \mu c\gamma = 0$ ..... (2).

The equation of the locus (Q) of the intersection of (1), (2) will therefore be $a\beta \pm b\alpha = 0$ ..... (3),

which is that of two straight lines (CD), passing through C, and cutting AB internally or externally in D, so that $AD : DB = AC^2 : BC^2$.

When ACB is a right angle, then $a = c\cos B$, $b = c\cos A$, and (3) becomes $a\cos A \pm \beta\cos B = 0$; whence, in this case, the internal line CQD is perpendicular to AB.

Or, by Pure Geometry. Let AF, BE meet BC, AC in R, S; then we have $AQ : AE = QR : RB$, $SQ : SA = BQ : BF$, $AE : AC = BF : BC$; therefore $AS.AC : BR.BC = AQ.QS : BQ.QR = \triangle AQS : \triangle BQR$.

Again, $AC : AS = \triangle AQC : \triangle AQS$, and $BR : BC = \triangle BQR : \triangle BQC$; therefore $AC^2 : BC^2 = \triangle AQC : \triangle BQC = AD : DB$.

Hence the locus of Q is a straight line passing through C and cutting AB in D, so that $AD : DB = AC^2 : BC^2$;

and when ACB is a right angle, and AE, BF proportional to AC, BC (in magnitude), then CQD will be perpendicular to AB.

2. When AE, BF are taken proportional to the *opposite* sides of the triangle, the equation (3) of the locus becomes $a\alpha \pm b\beta = 0$, which proves the second part of the theorem.

4354. (Proposed by R. TUCKER, M.A.)—Solve the equations

$$-x^2 + xy + xz = a = 4 \quad \ldots\ldots (1),$$
$$-y^2 + xy + yz = b = -20 \quad \ldots\ldots (2),$$
$$-z^2 + xz + yz = c = -8 \quad \ldots\ldots (3).$$

I. *Solution by* W. J. GREENFIELD, B.A.

(2)−(1) gives $(x-y)(x+y-z) = -(a-b)$ ..... (4).

(4) ÷ (3) gives $\frac{x-y}{z} = -\frac{a-b}{c}$;

similarly $\frac{x-z}{y} = -\frac{a-c}{b}$; $\frac{y-z}{x} = -\frac{b-c}{a}$ ..... (5).

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Substituting in (1) for $x-y$, we have

$$x\left(z+\frac{a-b}{c}z\right)=a, \text{ therefore } xz=\frac{bc}{a-b+c};$$

similarly $$xy=\frac{ab}{a+b-c}; \quad yz=\frac{bc}{b+c-a}.$$

Hence $x^2=\frac{a^2(b+c-a)}{(c+a-b)(a+b-c)}$, $y^2=\frac{b^2(c+a-b)}{(a+b-c)(c+a-b)}$, z^2 = &c.

In the particular case, $x=\pm 2$, $y=\pm 5$, $z=\mp 1$.

II. *Solution by* C. Leudesdorf.

(2) + (3) gives $b+c=xy+xz-(y-z)^2=a+x^2-(y-z)^2$ by (1);

therefore $b+c-a=(x-y+z)(x+z-y)$; and by (1) $a=x(y+z-x)$.

Hence, if $$R\equiv(y+z-x)(z+x-y)(x+y-z),$$

we have $a(b+c-a)=xR$, $b(c+a-b)=yR$, $c(a+b-c)=zR$;

therefore $(y+z-x)R=2bc+a^2-b^2-c^2=(-b+c+a)(-c+a+b)$,

with two similar equations; multiplying these together, we have

$$R.R^3=(b+c-a)^2(c+a-b)^2(a+b-c)^2;$$

therefore $$\pm R=\left\{(b+c-a)(c+a-b)(a+b-c)\right\}^{\frac{1}{2}}.$$

In the example given, $b+c-a=-32$, $c+a-b=+16$, $a+b-c=-8$;

therefore $R=\pm 64$, and $x=\pm 2$, $y=\pm 5$, $z=\mp 1$.

III. *Solution by the* Rev. G. H. Hopkins, M.A.

$$\frac{(1)-(2)}{(3)}, \text{ &c., gives } \frac{x-y}{z}=-\frac{a-b}{c}, \quad \frac{x-z}{y}=-\frac{a-c}{b};$$

therefore $$\frac{x}{a(b+c-a)}=\frac{y}{b(a-b+c)}=\frac{z}{c(a+b-c)}=\frac{1}{R} \text{ suppose.}$$

Substituting in (1) for the values of x, y, z in terms of λ, we have

$$R^2=(b+c-a)(a+c-b)(a+b-c);$$

and thence the same values as given in the preceding solutions.

Note on Question 4354. *By* Professor Cayley.

A question of simple algebra such as this, becomes more interesting when interpreted geometrically: thus, writing the equations in the form

$$-x^2+xy+xz=aw^2, \quad yx-y^2+yz=bw^2, \quad zx+zy-z^2=cw^2;$$

then, putting for shortness

$$\alpha=-a+b+c, \quad \beta=a-b+c, \quad \gamma=a+b-c,$$

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the solutions obtained are

$$x : y : z : w - \alpha a : b\beta : c\gamma : (a\beta\gamma)^{\frac{1}{2}},$$

$$x : y : z : w = \alpha a : b\beta : c\gamma : -(a\beta\gamma)^{\frac{1}{2}};$$

say these are $\{aa, b\beta, c\gamma, (a\beta\gamma)^{\frac{1}{2}}\}$ and $\{aa, b\beta, c\gamma, -(a\beta\gamma)^{\frac{1}{2}}\}$.

But the equations are also satisfied by

$$(x=0, y=z, w=0), \quad (y=0, z=x, w=0), \quad (z=0, x=y, w=0),$$

or what is the same thing (0, 1, 1, 0), (1, 0, 1, 0), (1, 1, 0, 0). The three equations represent quadric surfaces, each two of them intersecting in a proper quadric curve; and the three having in common 8 points; viz., these are made up of the first mentioned two points each once, and the last mentioned three points each twice: $2 + 3 \cdot 2, = 8$.

To verify this, observe that at each of the three points the tangent planes of the surfaces have a common line of intersection; this line is the tangent of the curve of intersection of any two of the surfaces, and the curve of intersection therefore touches the third surface; wherefore the point counts for two intersections. In fact, taking (X, Y, Z, W) as current coordinates, the equations of the tangent planes at the point (x, y, z, w) are

$$\begin{aligned} &X(2x-y-z) - Yx &&- Zx &&+ 2aWw = 0,\\ &-Xy &&+ Y(-x+2y-z) - Zy &&+ 2bWw = 0,\\ &-Xz &&- Yz &&+ Z(-x-y+2z) + 2cWw = 0, \end{aligned}$$

hence at the point (0, 1, 1, 0) these equations are

$$-2X = 0, \quad X + Y - Z = 0, \quad -X - Y + Z = 0,$$

which three planes meet in the line $X=0$, $Y-Z=0$; and similarly for the other two of the three points.

4355. (Proposed by H. H. Hudson, M.A.)—A curve is such that the radius vector makes half the angle with the normal that it does with a fixed line; find the chord of curvature through the pole.

Solution by B. Williamson, M.A.

Suppose θ the angle between the radius vector and the fixed line, and, to generalize the problem; let the angle between the radius vector and the normal be $m\theta$; then, if p be the perpendicular on the tangent, we have

$$p = r\cos m\theta, \text{ therefore } \frac{dp}{dr} = \cos m\theta - m \sin m\theta \frac{r\,d\theta}{dr},$$

but, evidently,

$$r\frac{d\theta}{dr} = -\cot m\theta;$$

hence

$$\frac{dp}{dr} = (m+1)\cos m\theta = \frac{(m+1)p}{r}.$$

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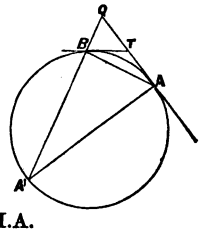
Again, if γ be the chord of curvature through the pole,

$$\frac{1}{\gamma} = \frac{dp}{pdr} = \frac{m+1}{r}, \text{ therefore } \gamma = \frac{r}{m+1}.$$

It may be remarked that the equation of the curve can be readily shown to be of the form $r^m = a^m \cos m\theta$.

4384. (Proposed by J. L. McKenzie.)—AA′ is a diameter of a conic, and Q any point in the tangent at A. The line A′Q cuts the conic in B; prove that the tangent at B bisects AQ.

I. *Solution by the* Rev. T. J. Sanderson, M.A.



Project the conic into a circle. Join AB; then TB=TA, and QBA is a right angle; therefore AT=TQ.

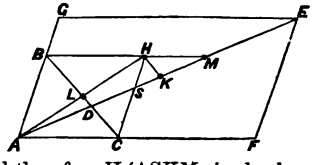
II. *Solution by* E. B. Elliott, M.A.

Let T be the intersection of the tangents at A and B. Join T to the centre C. Then CT bisects AB the polar of T. It also bisects AA′; therefore it is parallel to A′BQ; therefore it also bisects AQ.

4424. (Proposed by J. J. Walker, M.A.)—Forces P, Q act on a point A; and B, C, D are the points in which any transversal meets the lines of action of these forces and their resultant respectively; prove that

$$\frac{R}{AD} = \frac{P}{AB} + \frac{Q}{AC}.$$

I. *Solution by* H. Murphy.



Let AGEF represent the parallelogram of forces, BDC the transversal cutting the line of directions in B, D, and C; through B and C draw the parallels BM and CH, and HK parallel to BC; and join AH. In the triangle BHC, the base is bisected in L, and HK is parallel to the base; therefore H (BLCK) is a harmonic pencil, and therefore H (ASKM, is also har-

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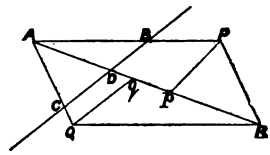
monic, and therefore $\frac{1}{AM} + \frac{1}{AS} = \frac{2}{AK} = \frac{1}{AD}$.

Hence $\frac{AE}{AM} + \frac{AE}{AS} = \frac{AE}{AD}$, or $\frac{P}{AB} + \frac{Q}{AC} = \frac{R}{AD}$.

II. *Solution by the* Rev. J. R. Wilson, M.A.

Draw Pp, Qq parallel to BCD. Then



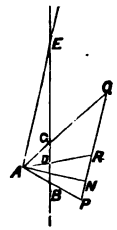
$$\frac{P}{AB} = \frac{Ap}{AD}, \quad \frac{Q}{AC} = \frac{Aq}{AD};$$

therefore $\frac{P}{AB} + \frac{Q}{AC} = \frac{Ap + Aq}{AD}$

$$= \frac{AR}{AD} = \frac{R}{AD}.$$

III. *Solution by* E. B. Elliott, M.A.

Along AB, AC, AD take AP, AQ, AR representing in magnitude the component forces P, Q and the semi-resultant $\frac{1}{2}$R respectively.



Through A draw AE parallel to PQ meeting BCD in E. Then because R bisects PQ, the pencil A (BCDE) and the range BCDE are harmonic; therefore

$$\frac{2}{ED} = \frac{1}{EB} + \frac{1}{EC},$$

$$\frac{2}{AD \sin EAD} = \frac{1}{AB \sin EAB} + \frac{1}{AC \sin EAC} \quad \ldots\ldots (1).$$

Now if AN be the perpendicular from A on PQ,

$$\sin EAD = \frac{AN}{AR} = \frac{AN}{\frac{1}{2}R}, \quad \sin EAB = \frac{AN}{AP} = \frac{AN}{P}, \quad \sin EAC = \frac{AN}{Q};$$

therefore by (1) we have $\frac{R}{AD} = \frac{P}{AB} + \frac{Q}{AC}$.

4180. (Proposed by M. Collins, LL.D.)—Prove that the equation $x^2 + D^m = (N^2 + D)y^2$ is always possible in rational numbers for x and y when N and D are rational, and m is an odd integer, and that x and y can be found in integers when N and D are integers.

Solution by A. M. Nash, B.A.

Let $m = 2n + 1$, then the equation may be written

$$x^2 - N^2y^2 - D(y^2 - D^{2n});$$

and this equation will be satisfied by any values of x and y which satisfy

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the two equations,

$$x + Ny = kD(y + D^n), \quad x - Ny = \frac{1}{k}(y - D^n) \quad \ldots\ldots\ldots (1),$$

k being indeterminate. Now, from (1) − (2) we have

$$y = \frac{(k^2D + 1)D^n}{2Nk + 1 - k^2D}, \text{ whence } x = \frac{\{N(k^2D - 1) + 2kD\}D^n}{2Nk + 1 - k^2D} \quad \ldots\ldots (2).$$

Since k is indeterminate, we can put it $= 0$, thus we get the values

$$x = -ND^n, \quad y = D^n;$$

and by splitting up the given equation differently, we can get

$$x = \pm ND^n, \quad y = \pm D^n;$$

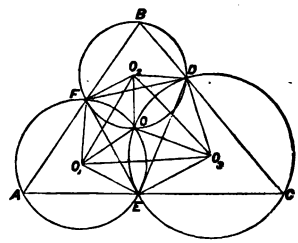
and these values are rational if N and D are rational, and also integral if N and D are integral.

4019. (Proposed by E. B. Seitz.)—The points D, E, F are taken at random in the sides BC, CA, AB of the triangle ABC respectively; show that if Δ be the area of the given triangle, and Δ_1 the average area of the triangle formed by joining the centres of the circles AEF, BFD, CDE; then

$$\Delta_1 = \tfrac{1}{24}(7 + \cot^2 A + \cot^2 B + \cot^2 C)\Delta.$$

Solution by Asher B. Evans, M.A.

Let O_1, O_2, O_3 be the centres of the circles AEF, BFD, CDE; O their common intersection; and r_1, r_2, r_3 their radii; $AF = x$, $BD = y$, $CE = z$. It is evident that O_1O_2, O_2O_3, O_3O_1 bisect the quadrilaterals OO_1FO_2, OO_2DO_3, OO_3EO_1; and that the hexagon $O_1FO_2DO_3E$ is double the triangle $O_1O_2O_3$. Therefore



$$2O_1O_2O_3 = DEF + O_1FE + O_2FD + O_3DE \ldots (1).$$

Since $2O_1FE = r_1^2 \sin 2A$, $2O_2FD = r_2^2 \sin 2B$, and $2O_3DE = r^3 \sin 2C$, equation (1) gives

$$O_1O_2O_3 - \tfrac{1}{2}DEF = \tfrac{1}{4}(r_1^2 \sin 2A + r_2^2 \sin 2B + r_3^2 \sin 2C) \ldots\ldots (2).$$

From the triangles AFE, O_1FE, we have

$$\left.\begin{aligned} (FE)^2 &= 4r_1^2 \sin^2 A = x^2 + (b-z)^2 - 2x(b-z)\cos A. \\ \text{Similarly } (FD)^2 &= 4r_2^2 \sin^2 B = y^2 + (c-x)^2 - 2y(c-x)\cos B, \\ \text{and } (DE)^2 &= 4r_3^2 \sin^2 C = z^2 + (a-y)^2 - 2z(a-y)\cos C. \end{aligned}\right\} \ldots\ldots (3).$$

From (2) and (3) we have

$$O_1O_2O_3 - \tfrac{1}{2}DEF = \frac{x^2 + (b-z)^2 - 2x(b-z)\cos A}{8\tan A} + \frac{y^2 + (c-x)^2 - 2y(c-x)\cos B}{8\tan B} + \frac{z^2 + (a-y)^2 - 2z(a-y)\cos C}{8\tan C} \ldots (4).$$

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Since the mean value of DEF is $\frac{1}{4}\Delta$, we have

$$\Delta_1 = \tfrac{1}{8}\Delta + \frac{1}{8abc}\int_0^c\int_0^a\int_0^b (4)\,dx\,dy\,dz = \tfrac{1}{8}\Delta + \frac{2b^2+2c^2-3bc\cos A}{48\tan A} + \frac{2a^2+2c^2-3ac\cos B}{48\tan B} + \frac{2a^2+2b^2-3ab\cos C}{48\tan C};$$

which, since $\cot A = \frac{b^2+c^2-a^2}{4\Delta}$, &c., gives the expression in the question.

4352. (Proposed by H. S. Monck.)—If δ be the deviation of a ray of light transmitted at the polarising angle of any medium, prove that the greatest difference of phase between the two components of a ray of polarised light which can be produced by total reflexion at the surface of the same medium is 2δ.

Solution by the Proposer.

The accelerations of the phases of the two components by total reflexion being 2θ and 2ϕ, we obtain the following values for $\tan\theta$ and $\tan\phi$ on Fresnel's Hypothesis. (See Airy. Article 133.)

$$\tan\theta = \frac{(\mu^2\sin^2 i - 1)^{\frac{1}{2}}}{\mu\cos i}, \quad \tan\phi = \frac{\mu\,(\mu^2\sin^2 i - 1)^{\frac{1}{2}}}{\cos i},$$

whence $\tan\phi = \mu^2\tan\theta$ and $\tan(\phi-\theta) = \frac{(\mu^2-1)\tan\theta}{1+\mu^2\tan^2\theta}$.

If we differentiate this expression, and make the differential coefficient $= 0$ for a maximum, we obtain

$$2\mu^2(\mu^2-1)\tan^2\theta\sec^2\theta - (\mu^2-1)\tan^2\theta(1+\mu^2\tan^2\theta) = 0;$$

therefore $2\mu^2\tan^2\theta = 1+\mu^2\tan^2\theta$, therefore $\mu^2\tan^2\theta = 1$,

hence we have $\tan\theta = \frac{1}{\mu}$, and $\tan\phi = \mu^2\tan\theta = \mu$.

θ and ϕ therefore are respectively equal to the angles of refraction and incidence at the polarising angle of the medium; and therefore $\phi-\theta$ (= half the phase-difference) is equal to $i-r$ the deviation of the ray transmitted at the polarising angle $\left(\tan i = \mu\tan r = \frac{1}{\mu}\right)$.

4422. (Proposed by W. B. Davis, B.A.)—Prove (1) that the sum of d odd squares is of the form $8b+d$; and (2) ascertain whether the converse is true.

I. *Solution by the* Rev. J. R. Wilson, M.A.

Let there be d odd squares $(2n+1)^2$, $(2n'+1)^2$, $(2n''+1)^2$, &c. Then

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their sum is $4\{n(n+1)+n'(n'+1)+n''(n''+1)+\&c.\}+d$,
which is of the form $8b+d$ since each of the terms within the bracket is divisible by 2.

II. *Solution by* HUGH MURPHY.

1. Every odd number is of the form $2n+1$, hence every square number is of the form $4n(n+1)+1$; the factors are evidently divisible by 8, therefore every square is of the form $8n+1$; hence d squares $\equiv 8dn+d = 8b+d$.

III. *Solution by* H. S. MONCK.

2. The converse is not true. It would imply that $8b+1$ was always a square number, and $8b+2$ always the sum of two odd squares. Neither statement is true, as may be seen by the numbers $17 = 2\times 8+1$ and $42 = 8\times 5+2$, which latter is not the sum of two odd squares.

I think, however, that the converse is true of the higher numbers. I have not been able to find an exception to it for $8b+3 = 3$ square numbers. In fact, the condition that $8b+1$ should be a square number is $b = \frac{1}{2}n(n+1)$, where n is any integer. $8b+2$ will be a sum of two square numbers on the same condition (since 1 is an odd square), and also if $b = \frac{1}{2}n(n+1)+\frac{1}{2}m(m+1)$, where n and m are any integers; and $8b+3$ will be the sum of three odd squares if either of the foregoing conditions be fulfilled, or if $b = \frac{1}{2}n(n+1)+\frac{1}{2}m(m+1)+\frac{1}{2}p(p+1)$, with any values for n, m, and p. I have not yet found a number which cannot be made up in one of these ways, but I have been equally unable to prove that every number must be capable of being so made up.

If the theorem holds for $8b+d$, it must evidently hold for $8b+(d+1)$, as 1 is an odd square; hence for all higher values of d.

4196. (Proposed by Professor TOWNSEND, F.R.S.)—If a conic have triple contact with a tricuspidal quartic, show that—

(a) The three quasi-normals at the three points of contact, with respect to the extremities of the bitangent chord, are concurrent.

(b) The three second tangents, which, with the three at the points of contact, divide the bitangent chord harmonically, are also concurrent.

(c) The line connecting the two points of concurrence passes through the intersection of the three cuspidal tangents, and is divided there and by the bitangent chord in the constant anharmonic ratio of $-3:1$.

Solution by the PROPOSER.

These three properties follow at once by projection from the corresponding three for the three-cusped hypocycloid, viz., that if a conic have triple contact with the curve—(a) the three normals at the three points of contact are concurrent; (b) the three second tangents at right angles to those at the three points of contact are also concurrent; (c) the line connecting the two points of concurrence passes through the cuspidal centre, and is there divided internally in the constant ratio of 3 : 1. (See *Reprint*, Vol. IV., p. 16.)

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4397. (Proposed by S. Tebay, B.A.)—A circle is drawn at random within a given circle; show that (1) the probability that it contains the centroid of the remaining figure is $\frac{8}{105}$; and (2) the analogous probability for a sphere is $\frac{81}{1820}$.

Solution by the Proposer.

1. Let the radius of the given circle be unity, r the radius of any other circle within it, y the distance of their centres, and x the distance of the centre of gravity of the remaining figure from the centre.

Then $x = \frac{r^2 y}{1-r^2}$. Let $x = r - y$; then $y = r - r^3$; and the probability required is $\int_0^1 \int_0^y dy = \int_0^1 dr\,(r - r^3) = \frac{8}{105}$.

2. Similarly for the sphere it may be found that the analogous probability is $\frac{81}{1820}$.

4350. (Proposed by C. W. Merrifield, F.R.S.)—Of all convex polyhedra having their dihedral lengths and their edges all equal, there are, besides the four Platonic or regular solids, only two which can be inscribed, and two which can be circumscribed, to a sphere.

Solution by the Proposer.

Take the latter case first, and consider any two tangent planes. The plane through the centre and points of contact meets their intersection at right angles and at a point equi-distant from the point of contact. Therefore a circle can be inscribed in each facet, which must consequently be either a regular polygon or a rhombus. We need not consider hexagonal facets, because these must have plane angles of 120 degrees or more. The first case leads to the Platonic solids. In the second case, we have supplementary angles adjacent to each edge, and the grouping must be by m angles or n supplementary angles to form a corner.

Of the obtuse angles, it is clear that three always must go to form a corner. It only remains to discuss the possible sets of the other corners.

If m plane angles, each $= 2\theta$, go to make a corner, the dihedral angle 2ϕ will be given by $\cos\frac{\pi}{m} = \sin\phi\cos\theta$.

Now, since three obtuse angles are to give the same dihedral angle as m acute angles, each of which is their supplement, we have

$$\cos\frac{\pi}{m} = \sin\phi\cos\theta \ (\theta \text{ an acute angle}),$$

$$\cos\frac{\pi}{3} = \sin\phi\sin\theta,$$

therefore $2\tan\theta\cos\frac{\pi}{m} = 1$.

Let the number of facets of the solid be f. Then, since the facets are

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quadrilateral, the number of edges is $2f$, and of plane angles $4f$. Also, let there be a corners of three obtuse angles, and b of m acute angles. Then, by mere counting, $3a+mb=4f$, and $a+b+f=2f+2$.

Eliminating f, we have

$$3a+mb=4(a+b-2), \text{ or } 3a-mb=2(a-2b+4).$$

Next, let us consider the sphere as divided into polygons about the corners by planes through its centre and through the points of contact, viz., a equilateral triangles, and b m-sided regular polygons. Since these polygons will be the polars of the corners of the solid circumscribed to the sphere, the dihedral angles of the triangles will be each 2θ, and of the polygons $\pi-2\theta$. The spherical area of each triangle will be $6\theta-\pi$ and of each polygon $m(\pi-2\theta)-(m-2)\pi=2\pi-2m\theta$. Their sum, which is to equal the area of the sphere, will therefore be

$$a(6\theta-\pi)+b(2\pi-2m\theta)=4\pi, \text{ or } 2(3a-mb)\theta=(a-2b+4)\pi.$$

But since $$3a-mb=2(a-2b+4),$$

we must either have $4\theta=\pi$, which gives the cube, or

$$3a-mb=2(a-2b+4)=0.$$

The two last give $a=\frac{1}{3}mb=2b-4$, $(6-m)b=12$.

Since a, m and b are both positive and integral, it is evident that m cannot exceed 5, and it cannot be less than 3.

If $m=3$, we have $a=b=4$, which gives the cube again.

If $m=4$, we have $b=6$, $a=8$, $f=12$, $\tan\theta=\frac{1}{\sqrt{2}}$,

which gives the right rhombic dodecahedron.

If $m=5$, we have

$$b=12,\ a=20,\ f=30,\ 2\tan\theta\cos\frac{\pi}{5}=1,\ \tan\theta=\sin 18°.$$

This gives a solid of 30 rhombic facets, which is that circumscribed to a sphere passing through the middle points of the edges of a regular dodecahedron or icosahedron; its facets passing through the edges of that solid.

The inscribed case evidently corresponds to the circumscribed case, and the solids obtained are—

For $m=4$, by joining the middle points of adjacent edges of the cube or octohedron. This gives a solid with 12 corners and 14 facets, 6 of which are squares, and 8 equilateral triangles.

For $m=5$, by joining the middle points of adjacent edges of the regular dodecahedron or icosahedron. This gives a solid with 30 corners and 32 facets, 12 regular pentagons, and 20 equilateral triangles.

Of these solids, the only one of importance is the right rhombic dodecahedron, a number of which will fill space without interstices. It is formed by drawing tangent planes at the points of contact of *round shot in pile*, whether in triangular or quadrilateral pile.

4421. (Proposed by B. Williamson, M.A.)—If any straight line drawn through the pole of a spiral of Archimedes, meet two consecutive branches of the curve in the points A, B, and A′, B′ respectively; prove

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that the area between the line and the two branches is half that of the ellipse whose semi-axes are AA′ and BA′.

I. *Solution by the* PROPOSER.

The sectorial area between two consecutive branches is $\frac{1}{2}\int (r_2^2 - r_1^2)\, d\theta$ taken between suitable limits moreover, $r = a\theta$ be the equation curve, we have $r_2 = r_1 + 2a\pi = r_1 + c$,

if $c = 2a\pi$; therefore, $r_2^2 - r_1^2 = 2r_1 c + c^2$,

and $\frac{1}{2}\int (r_2^2 - r_1^2)\, d\theta = c\int a\theta\, d\theta + \frac{1}{2}c^2 \int d\theta = \frac{1}{2}c\,(\beta - \alpha)(a\beta + a\alpha + c)$,

α, β denoting the limiting values of θ. Now let $\beta - \alpha = \pi$, and the result in question follows immediately.

II. *Solution by the* Rev. J. R. WILSON, M.A.

Let θ_1, θ_2 be values of θ corresponding to SA′ and SB′, S being the pole. Then the expression for the area included between SA′, SB′ and the curve $r = a\theta$ is (TODHUNTER'S *Integral Calculus*, Art. 153)

$$\tfrac{1}{2}a^2 \left\{ 2\pi\,(\theta_2^2 - \theta_1^2) + 4\pi^2\,(\theta_2 - \theta_1) \right\}.$$

Putting $\theta_2 = \theta_1 + \pi$, this reduces to $\pi^2 a^2\,(3\pi + 2\theta_1)$.

Again $AA' = SA' - SA = 2a\pi$, and $BA' = SB + SA' = 2a\theta_1 + 3a\pi$.

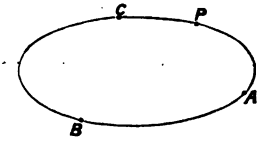
Therefore the area of the ellipse whose semi-axes are AA′ and BA′ is

$$\pi \,.\, 2a\pi\,(2a\theta_1 + 3a\pi) = 2\pi^2 a^2\,(3\pi + 2\theta_1).$$

Hence the result stated in the question.

4396. (Proposed by J. J. WALKER, M.A.)—Find at what points on a conic the angle between the tangent and the line drawn to a fixed point on the conic is a maximum or minimum.

I. *Solution by* F. D. THOMSON, M.A.



In the neighbourhood of such a point, two consecutive tangents make equal angles with the two chords drawn to the fixed point. Hence the circle of curvature passes through the fixed point. But if ϕ, ψ be the excentric angles at the point of maximum or minimum and at the fixed point, $3\phi + \psi = 2n\pi$, or $\phi = \frac{2}{3}n\pi - \frac{1}{3}\psi$. This gives the three points determined by $-\frac{1}{3}\psi$, $\frac{1}{3}(2\pi - \psi)$, $\frac{1}{3}\,4\pi - \psi)$. Also the angle between the tangent and the chord is a minimum at the fixed point, and the other portions will be alternately maxima and minima. Thus, in the figure, if P be the fixed point, A and C will be points of maxima, and B of a minimum.

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II. *Solution by the* PROPOSER.

Writing the equation to the conic (referred to its axis and the tangent at the vertex) $ax^2+cy^2+2dx=0$ ..... (1),
the square of sine of the angle referred to is readily found to be equal to

$$\frac{-cd^2x}{\{(a-c)(ax^2+2dx)+d^2\}\{(a-c)x+2d\}} \quad \text{..... (2),}$$

and this expression will be a maximum or minimum when x satisfies the equation $a(a-c)^2x^3+d(a-c)(2a-c)x^2-d^3=0$.
It may easily be verified that one root of this equation is

$$x=\frac{-d}{a-c} \quad \text{..... (3),}$$

which gives 1 as the value of (2), and that the other two roots are given by the quadratic $(a-c)(ax+d)x-d^2=0$ ..... (4).
In the case of the parabola, $a=0$, and (3) gives $x=\frac{d}{c}=-\frac{1}{4}p$, an impossible value; but (4) reduces to $x=-\frac{d}{c}=\frac{1}{4}p$; and at this point the sine of the angle in question has its maximum value, viz. $\frac{1}{2}$.

4294. (Proposed by Professor WOLSTENHOLME, M.A.)—Prove that

$$2^n(x-1)^n-(n-1)2^{n-2}(x-1)^{n-2}(x+1)^2$$
$$+\frac{(n-2)(n-3)}{\underline{|2}}(x-1)^{n-4}(x+1)^4-\text{.....} \text{ to } \frac{n+1}{2} \text{ or } \frac{n}{2}+1 \text{ terms,}$$
$$\equiv(n+1)\left(x^n-\frac{2(n+1)2n}{\underline{|3}}x^{n-1}+\frac{2(n+1)\cdot 2n\cdot 2(n-1)\cdot 2(n-2)}{\underline{|5}}x^{n-2}-\right.$$
$$\text{..... to } (n+1) \text{ terms,}$$

n being any integer.

Solution by R. TUCKER, M.A.

In the Proposer's *Book of Mathematical Problems*, Ex. 169, we have

$$(p+q)^n-(n-1)(p+q)^{n-2}pq+\frac{(n-2)(n-3)}{1\cdot 2}(p+q)^{n-4}p^2q^2-\text{.....}$$
$$\equiv\frac{p^{n+1}-q^{n+1}}{p-q} \quad \text{..... (A).}$$

If now we assume $p+q=2(x-1)$, $pq=(x+1)^2$ ..... (B),
we get the sinister member in the proposed question.
From (B) we have

$$p=\{x^{\frac{1}{2}}+(-1)^{\frac{1}{2}}\}^2, \quad q=\{x^{\frac{1}{2}}-(-1)^{\frac{1}{2}}\}^2,$$

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and $p^{n+1}-q^{n+1} = (x^{\frac{1}{2}}+i)^{2n+2} - (x^{\frac{1}{2}}-i)^{2n+2}$

$$= 4i(n+1)\left[x^{\frac{1}{2}(2n+1)} - \frac{(2n+1)\,2n}{\lfloor 3}\, x^{\frac{1}{2}(2n-1)} + \ldots\ldots\right]$$

$$p-q = 4i\,x^{\frac{1}{2}};$$

hence the dexter-member of (A) in this case

$$= (n+1)\left[x^n - \frac{(2n+1)(2n)}{\lfloor 3}\, x^{n-1} + \&c.\right],$$

whence the truth of the identity is seen. We may prove (A) thus

$$\frac{1}{1-(p+q)\,y+pqy^2} = \frac{1}{(1-py)(1-qy)} = \frac{1}{p-q}\left(\frac{p}{1-py} - \frac{q}{1-qy}\right).$$

Now equate the coefficients of y^n, and we get

$$\frac{p^{n+1}-q^{n+1}}{p-q} = \text{coefficient of } y^n \text{ in } \left[1-y\left\{(p+q)-pqy\right\}\right]^{-1}$$

$$= (p+q)^n - (n-1)(p+q)^{n-2}pq + \&c.$$

4386. (Proposed by J. J. Sylvester, F.R.S.)—Given a cubic equation in x and a quadratic equation in y, express under the form of a determinant the left-hand side of the equation, whose roots are $lx+my$.

I. *Solution by* J. L. McKenzie.

Let $\phi(x) = 0$ and $\psi(y) = 0$ be any two equations in x and y, and

$$f(u) \equiv u^t + p_1 u^{t-1} + p_2 u^{t-2} + \ldots + p_t = 0,$$

the equation whose roots are of the form $lx+my$. Let α_r, β_r, s_r be the sums of the rth powers of the roots of these three equations, and $a_1, a_2, a_3, \ldots\ldots$ $b_1, b_2, \ldots\ldots$ the roots of $\phi(x)$ and $\psi(y)$. Thus

$$\begin{aligned}
s_r = {} & (la_1+mb_1)^r + (la_2+mb_1)^r + \ldots\ldots \\
& + (la_1+mb_2)^r + (la_2+mb_2)^r + \ldots\ldots \\
& \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\
= {} & l^r a_1^r + rlma_1^{r-1}b_1 + \frac{r(r-1)}{\lfloor 2}\, l^{r-2}m^2 a_1^{r-2}b_1^2 + \ldots\ldots \\
& + l^r a_2^r + rlma_2^{r-1}b_1 + \frac{r(r-1)}{\lfloor 2}\, l^{r-2}m^2 a_2^{r-2}b_1^2 + \ldots\ldots \\
& \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\
& + l^r a_1^r + rlma_1^{r-1}b_2 + \frac{r(r-1)}{\lfloor 2}\, l^{r-2}m^2 a_1^{r-2}b_2^2 + \ldots\ldots \\
& \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots
\end{aligned}$$

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Adding the columns, we have

$$s_r = l^r a_r \beta_0 + r l^{r-1} m a_{r-1} \beta_1 + \frac{r(r-1)}{\lfloor 2} l^{r-2} m^2 a_{r-2} \beta_2 + \ldots\ldots$$

$$= (la + m\beta)^r,$$

if we remember, in the expansion, to attach the proper index of a and β in each term, not as an index, but as a suffix. It must also be remembered that a_0 and β_0 are not units, but are equal to the degrees of ϕ and ψ respectively, and therefore cannot be neglected in the terms in which they occur as factors.

It is easily seen that if any number of equations $\phi(x) = 0$, $\psi(y) = 0$, $\chi(z) = 0$, &c., are given, the sum of the rth powers of the roots of the equation whose roots are $lx + my + nz + \ldots\ldots$ is given by the formula

$$s_r = (la + m\beta + n\gamma + \ldots\ldots)^r,$$

the expansion being performed by the multinomial theorem, subject to the same conditions as before.

The values of s_1, s_2, s_3, &c. being thus known, it is easy to express $f(u)$ in the form of a determinant. For, by Newton's well-known theorem,

$$\begin{aligned}
s_1 + p_1 &= 0,\\
s_2 + p_1 s_1 + 2p_2 &= 0,\\
s_3 + p_1 s_2 + p_2 s_3 + 3p_3 &= 0,\\
\ldots\quad \ldots\quad \ldots\quad \ldots\quad &\ldots\\
s_t + p_1 s_{t-1} + p_2 s_{t-2} \ldots\ldots + tp_t &= 0.
\end{aligned}$$

Eliminating $1, p_1, p_2, \ldots p_t$, between these equations and $f(u) = 0$, we have

$$\begin{vmatrix}
u & u^{t-1} & u^{t-2} & \ldots\ldots & 1\\
s_1 & 1 & 0 & \ldots\ldots & 0\\
s_2 & s_1 & 2 & \ldots\ldots & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
s & s_{t-1} & s_{t-2} & \ldots\ldots & t
\end{vmatrix} = 0.$$

It is seen, by considering the coefficient of u, that the only extraneous factor in this equation is $\lfloor t$. The value of t is the product of the degrees of ϕ, ψ, χ, &c.

II. *Solution by the* PROPOSER.

This amounts to finding the resultant of $(x, 1)^3$; $(y, 1)^2$; $lx + my + \rho$; in respect to x, y, 1. Multiplying the equations respectively by 1, y; 1, x, x^2; 1, y, x, xy, x^2, x^2y, so as to obtain 11 equations, it will be found that there are 11 arguments to be eliminated; viz., 1, x, y, x^2, xy, y^2, x^2, xy^2, y^3, x^2y^2, xy^3. Consequently the resultant can be obtained by the Dialytic process, and will be of the 2nd, 3rd, 6th degree in the coefficients of the three equations respectively, as it ought to be. We thus obtain $F(\rho) = 0$, where $F(\rho)$ is exhibited under the form of a determinant, as was to be found.

4394. (Proposed by T. T. WILKINSON, F.R.A.S.)—When two points are given in the circumference of a given circle; another circle may be

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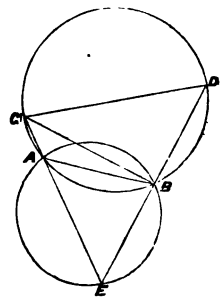
found, given in position; such, that if any two lines be drawn from the given points to intersect in its circumference and meet the given circle in two points, the line joining these points will be of constant length.

Solution by CHRISTINE LADD; H. MURPHY; *and others.*

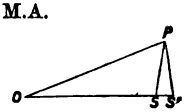
Because the line CD is given, and the points A and B are given, the angles CBD and ACB are given; therefore their difference AEB is given; and hence the required circle is one drawn on AB so as to contain a given angle.



4432. (Proposed by N'IMPORTE.)—An indefinite number of ellipses are drawn with an endless string, of length $2a$. One focus is fixed and the other moves on a straight line; find the equation to the locus of the intersections of the ellipses.

I. *Solution by the* Rev. J. R. WILSON, M.A.

Taking O, the fixed focus, as origin; S, S′ to be two positions of the other focus, and P a point on the locus, we have $OP + SP = OP + S'P$



therefore $SP = S'P$.

Therefore, ultimately, SP is perpendicular to OS. Let $OS = x$, $SP = y$,

therefore $x^2 + y^2 = OP^2 = (2a - SP)^2 = 4a^2 - 4ay + y^2$,

therefore $x^2 = 4a\,(a - y)$.

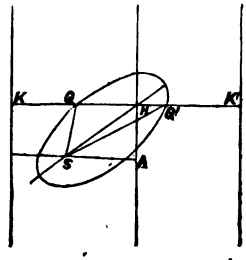
Hence the locus is a parabola whose focus is O, whose axis is perpendicular to OS, and whose latus rectum is $4a$.

II. *Solution by* Professor WOLSTENHOLME, M.A.

Let S be the given focus, AH the given straight line, and draw to it two parallels each at distance $2a$, and through H draw KHK′ perpendicular to these two. Then if S, H be foci of one of the ellipses, and this meet KK′ in QQ′, we have

$SQ' + Q'H = 2a = HK'$, or $SQ' = Q'K'$,

so $SQ = QK$; hence Q, Q′ lie on two parabolas of which S is the focus, and the two parallels to the given straight line the directrices; also the tangents to these at Q, Q′ will touch the ellipse, since they make equal angles with SQ, QH; SQ′, Q′H respectively.



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These parabolas are the envelope of the system of ellipses; and also of the system of hyperbolas having S for focus, the given straight line for the locus of the second focus, and major axis equal to $2a$.

Analytically, if SA be the prime radius, ASH $= \alpha$, SA $= 2c$, then $e = \frac{\text{SH}}{2a} = \frac{c}{a \cos \alpha}$, and the equation of one of the ellipses is

$$\frac{a(1-e^2)}{r} = 1 - e\cos(\theta - \alpha),$$

or
$$\frac{a^2\cos^2\alpha - c^2}{r} = a\cos^2\alpha - c\cos\alpha\cos(\theta-\alpha),$$

or
$$\frac{a^2(1+\cos 2\alpha) - 2c^2}{r} = a(1+\cos 2\alpha) - c\left\{\cos\theta + \cos(\theta - 2\alpha)\right\},$$

$$\cos 2\alpha\left(\frac{a^2}{r} - a + c\cos\theta\right) + \sin 2\alpha \cdot c\sin\theta = a - c\cos\theta - \frac{a^2-2c^2}{r},$$

and the envelope is

$$\left(\frac{a^2}{r} - a + c\cos\theta\right)^2 + c^2\sin^2\theta = \left(a - c\cos\theta - \frac{a^2-2c^2}{r}\right)^2,$$

or
$$c^2\sin^2\theta = \left(a - c\cos\theta - \frac{a^2-2c^2}{r}\right)^2 - \left(\frac{a^2}{r} - a + \cos\theta\right)^2$$

$$= \left(2a - 2c\cos\theta - \frac{2a^2-2c^2}{r}\right)\left(\frac{2c^2}{r}\right),$$

or
$$r^2\sin^2\theta = 4\left\{r(a - c\cos\theta) - (a^2 - c^2)\right\},$$

or
$$r^2 - 4ar + 4a^2 = r^2\cos^2\theta - 4cr\cos\theta + 4c^2;$$

therefore
$$r - 2a = \pm(r\cos\theta - 2c), \text{ and } r = \frac{2(a \pm c)}{1 \pm \cos\theta};$$

the same result as before. I must own to having found the envelope in this way first, the result of course suggesting the geometrical proof, which is much the better. The particular case when $c = 0$ is one of the problems in my book:—"Given a focus and the length and direction of the major axis, the envelope is two parabolas;" but this generalizing of it I never saw before. The reciprocal of this with respect to S is as follows:—A circle is described such that the polar of a fixed point S with respect to it always passes through a fixed point; also LL′ being the diameter through S, $\frac{1}{\text{SL}} - \frac{1}{\text{SL}'}$ is equal to a given constant; the envelope is two fixed circles touching each other at O. The locus of the centre of this circle being a conic, that of the directrix corresponding to S in the original will also be a conic.

The equation of the circle will be

$$r^2 - 2pr\cos(\theta - \alpha) + p^2 - q^2 = 0,$$

where
$$p(p - a\cos\alpha) = q^2, \text{ and } \frac{1}{p-q} - \frac{1}{p+q} = \frac{2}{b},$$

so that the equation may be written, with α as the only parameter,

$$r^2(-b^2 - a^2\cos^2\alpha) - 2ab^2 r\cos\theta\cos(\theta-\alpha) + a^2b^2\cos^2\alpha = 0,$$

or in either of the two forms

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$$r^2(2b^2-a^2)-2ab^2r\cos\theta-a^2b^2$$
$$=(a^2r^2+2ab^2r\cos\theta-a^2b^2)\cos 2\alpha+2ab^2r\sin\theta\sin\alpha \quad \ldots\ldots (1),$$
$$r^2b^2\tan^2\alpha-2ab^2r\sin\theta\tan\alpha+(b^2-a^2)r^2-2ab^2r\cos\theta+a^2b^2=0 \quad \ldots\ldots (2),$$

whence the envelope, according to (1), is

$$\left\{r^2(2b^2-a^2)-2ab^2r\cos\theta+a^2b^2\right\}^2-(a^2r^2+2ab^2r\cos\theta-a^2b^2)^2$$
$$=4a^2b^4r^2\sin^2\theta,$$

and to (2) is $a^2b^4r^2\sin^2\theta=b^2r^2\left\{(b^2-a^2)r^2-2ab^2r\cos\theta+a^2b^2\right\}$,

which in either case reduce to $r=\dfrac{ab}{b\pm a}\cos\theta$.

The coordinates of the centre are (p, α), whence its equation is

$$\frac{r}{b}=\frac{ab\cos\theta}{b^2-a^2\cos^2\theta},$$

or in rectangular coordinates $b^2(x^2+y^2)-a^2x^2=ab^2x$,

a conic of which S is a vertex, and major axis an arithmetic mean between the diameter of the two circles, or its major axis is the straight line joining S to the middle point of the distance between the two centres. Hence the envelope of the directrix in the original will be a parabola.

Its equation will be $r\cos(\theta-\alpha)=-\dfrac{a(1-e^2)}{e}=+\dfrac{a(c^2-a^2\cos^2\alpha)}{ac\cos\alpha}$,

or $$r\cos\theta+r\sin\theta\tan\alpha=\frac{c^2-a^2}{c}+c\tan^2\alpha,$$

and the envelope is

$$r^2\sin^2\theta=4c\left(\frac{c^2-a^2}{c}-r\cos\theta\right), \text{ or } y^2=4c\left(\frac{c^2-a^2}{c}-x\right),$$

the point of contact being $y=2c\tan\alpha=AH$.

If HQ be drawn parallel to AS to meet the directrix (of S) in Q, it may easily be shown geometrically that Q lies on a fixed parabola to which the directrix is tangent.

The equation of the other directrix is

$$r\cos(\theta-\alpha)=\frac{a(1+e^2)}{e}=\frac{c^2+a^2\cos^2\alpha}{c\cos\alpha},$$

and since this only differs from the other in the sign of a^2, its envelope will be the parabola $$y^2=4c\left(\frac{a^2+c^2}{c}-x\right),$$

the point of contact being also on the straight line through H parallel to SA.

The latus-rectum through H, and the minor axis, will, by the original construction, obviously envelope parabolas focus S.

4173. (Proposed by Professor TOWNSEND, F.R.S.)—If the quasi-evolute of a tricuspidal quartic curve be taken with respect either to the

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two points of contact, real or imaginary, of its double tangent, or to a pair of its cusps, real or imaginary; show that it will be another tricuspidal quartic, having the same cuspidal centre and tangents, and the same pair either of bitangent points or of cusps of reference, as the original.

Solution by the PROPOSER.

For both properties being true, the former for the three-cusped hypocycloid and the latter for the cardioid, for both of which the quasi for the points of reference coincides with the actual evolute, the bitangent points for the former and the imaginary cusps for the latter being the circular points at infinity, they are therefore true by projection for all tricuspidal quartics; and therefore, &c.

That, by a real projection, any tricuspidal quartic may be projected either into a regular tricusp or into a cardioid, according as its cusps are all real or two of them imaginary, is evident from the consideration that the conic, always real, which touches the curve at its three intersections with its three cuspidal tangents, and which passes through its two points of contact with its bitangent, may always be projected into a circle having for centre the projection of the point of concurrence of the three cuspidal tangents, in which case the curve itself would manifestly be projected into the regular tricusp or cardioid according to its class.

4378. (Proposed by W. H. H. HUDSON, M.A.)—If A, A′ are different powers of the prime factor a, and B, B′ of another prime factor b, and if AB, AB′, A′B, A′B′, have n, m, p, q divisors, prove that abABA′B′ has $(n+m+p+q)$ divisors.

I. *Solution by* E. B. ELLIOTT, M.A.; A. B. EVANS, M.A.; *and others.*

Let α, α' be the exponents of a in A and A′ respectively; β, β' those of b in B and B′. Then in abABA′B′ the exponent of a is $1+\alpha+\alpha'$, and that of b is $1+\beta+\beta'$. Therefore number of divisors of abABA′B′

$$= (1+1+\alpha+\alpha')(1+1+\beta+\beta')$$
$$= (1+\alpha)(1+\alpha') + (1+\alpha)(1+\beta') + (1+\alpha')(1+\beta) + (1+\beta)(1+\beta')$$
$$= n+m+p+q.$$

II. *Solution by the* Rev. G. H. HOPKINS, M.A.

In a more extended form this theorem becomes: If $A_1, A_2, A_3, \ldots A_n$ are different powers of the prime factor A_0; $B_1, B_2, B_3, \ldots B_n$ of another prime factor B_0; .....; $H_1, H_2, H_3, \ldots H_n$ different powers of the prime factor H_0, and there are n such series, $N_1, N_2, N_3, \ldots N_n$ being the last; and if also $A_r . B_s . C_q \ldots N_e$ has $\Delta_{rsq\ldots e}$ divisors, then

$$A_0^{n-1} A_1 A_2 A_3 \ldots A_n B_0^{n-1} B_1 B_2 \ldots B_n \ldots N_0^{n-1} N_1 N_2 \ldots N_n$$

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has $\Sigma(\Delta_\phi)$, where ϕ can have any form similar to $rsq \dots e$, the different letters having any value from 1 to n.

Let $$A_1 = A_0^{\alpha_1}, \quad B_1 = B_0^{\beta_1}, \quad \dots\dots, \quad N_1 = N_0^{\nu_1},$$
$$A_2 = A_0^{\alpha_2}, \quad B_2 = B_0^{\beta_2}, \quad \dots\dots, \quad N_2 = N_0^{\nu_2},$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$A_n = A_0^{\alpha_n}, \quad B_n = B_0^{\beta_n}, \quad \dots\dots, \quad N_n = N_0^{\nu_n}.$$

Then $$A_r B_s C_q \dots N_e = A_0^{\alpha_r} B_0^{\beta_s} C_0^{\lambda_q} \dots N_0^{\nu_e},$$

which has $(\alpha_r + 1)(\beta_s + 1)(\gamma_q + 1) \dots (\nu_e + 1)$ divisors,

which equal $\Delta_{rsq \dots e}$. Therefore

$$A_0^{n-1} A_1 A_2 A_3 \dots A_n B_0^{n-1} B_1 B_2 \dots B_n \dots N_0^{n-1} N_1 N_2 \dots N_n$$
$$= A_0^{n-1+\alpha_1+\alpha_2+\alpha_3 \dots \alpha_n} B_0^{n-1+\beta_1+\beta_2+\beta_3 \dots +\beta_n} \dots\dots N_0^{n-1+\nu_1+\nu_2+\dots \nu_n},$$

which has for the number of its divisors

$$(n-1+\alpha_1+\alpha_2+\alpha_3+\dots\alpha_n+1)(n-1+\beta_1+\beta_2+\beta_3\dots+\beta_n+1)\dots$$
$$\dots(n-1+\nu_1+\nu_2\dots\nu_n+1),$$

or $$(\alpha_1+1+\alpha_2+1+\alpha_3+1\dots\alpha_n+1)(\beta_1+1+\beta_2+1+\dots\beta_n+1)\dots$$
$$\dots(\nu_1+1+\nu_2+1\dots\nu_n+1).$$

In the expansion of these by ordinary multiplication, treating α_1+1, β_k+1, &c., as single terms, we have

$$(\alpha_r+1)(\beta_s+1)(\gamma_q+1)\dots(\nu_e+1) + \text{similar terms},$$

which equals $\Sigma(\Delta_\phi)$.

4359. (Proposed by N'Importe.)—Prove the truth of the following construction for inscribing a square in a given triangle ABC: Draw BD perpendicular to AC; complete the rectangle BDCE, and then the square CEFG; then a line parallel to AC through the intersection (O) of AF and CE will give one side of the required square.

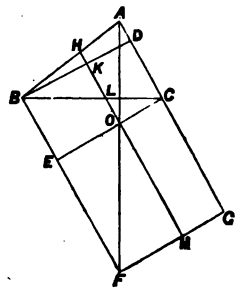
I. *Solution by* J. L. McKenzie.

From the similar triangles BHL and BAC,

we have $$\frac{BK}{BD} = \frac{HL}{AC};$$

but $$\frac{BK}{BD} = \frac{FM}{OM} = \frac{OC}{AC} = \frac{KD}{AC}.$$

Therefore HL = KD, and HL is evidently one side of an inscribed square.



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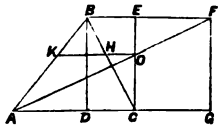
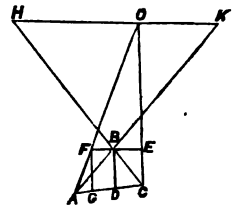
II. *Solution by* H. S. Monck.



(Fig. 1.)



(Fig. 2.)

Let OHK (Fig. 1) be the parallel to AC drawn through O; then, by similar triangles, we have

BF : KO = BA : KA = BC : HC = EC : OC; also BE : HO = EC : OC; hence EF : KH = EC : OC. But EF = EC, therefore KH = OC.

And if perpendiculars were drawn from K and H on AC, they would evidently be equal to OC. Hence KH is a side of the inscribed square.

Cor. 1.—In the same way we can inscribe in a triangle a rectangle having its sides in a given ratio. To do this, instead of making a square on CE, make a rectangle having its sides in the given ratio. Then the conclusion will be as follows:—

But EF : EC = m : n, therefore KH : OC = m : n.

Cor. 2.—Or we can inscribe in a triangle a parallelogram having a given angle, and having its sides in a given ratio. To do this, instead of drawing BD perpendicular to the base (AC) let it be drawn so as to make the given angle therewith. Then complete the parallelogram BDEC, and on EC form another parallelogram with its sides in the given ratio. The intersection of AF and CE gives the point as before.

Cor. 3.—If we construct the square on the other side of CE (Fig. 2), we obtain the escribed square. The proof proceeds as before, except that we have to add BF and BE instead of subtracting them.

4426. (Proposed by W. H. H. Hudson, M.A.)—A particle moves in a resisting medium, the law of resistance being expressed by kv^2; find what initial velocity would be reduced one-half in a unit of time. Of what dimensions is k?

Solution by G. S. Carr.

The equation of motion is $f = \frac{d^2x}{dt^2} = -kv^2$ ..... (1).

Let V be the initial velocity, and F the initial retarding force on a unit of mass. The value of F will depend upon the density of the medium as well as upon the magnitude of V.

From equation (1),

$$kt = \frac{1}{v} - \frac{1}{V},$$ [Tait and Steele's *Dynamics*, art. 215].

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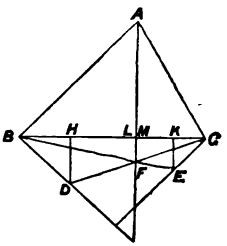
Putting $t = 1$ and $v = \frac{1}{2}V$, we find $V = k^{-1}$. But initially $V^2 = -Fk^{-1}$, therefore $V = -F$. That is, the velocity which will be reduced one half in a unit of time, must be equal to the acceleration which measures the force of resistance at the commencement. The absolute value of such velocity will be determined by the density of the medium. If a body be moved through the medium with increasing velocity until its momentum is equal to the resistance experienced, it will then, if left to itself, lose half of its velocity in one unit of time.

4293. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—At the ends B, C of the base of a triangle ABC, perpendiculars BD, CE are drawn to the sides AB, AC having a constant ratio to these sides; prove that the lines which join the ends of these perpendiculars with the opposite angles meet on the perpendicular from A on the base BC.

Solution by F. B. W. PHILLIPS; R. TUCKER, M.A.; M. COLLINS, LL.D.; H. MURPHY; *and others.*

Draw DH, EK, AL perpendicular to BC; then, by similar triangles, $\frac{DH}{EK} = \frac{BL}{CL}$; also BH = CK and BK = CH; therefore, if FM be the perpendicular from F on BC, we have



$$\frac{DH}{FM} = \frac{CH}{CM}, \quad \frac{EK}{FM} = \frac{BK}{BM};$$

hence $$\frac{DH}{EK} = \frac{BM}{CM} = \frac{BL}{CL};$$

i. e., M coincides with L, which proves the theorem.

4399. (Proposed by J. GRIFFITHS, M.A.)—Let A_1, B_1, C_1 be the vertices of a triangle inscribed in another given triangle ABC so as to be homologous with it; P any point on their axis of homology; X, Y, Z the points where the lines AP, BP, CP meet the sides BC, CA, AB. Show that the envelope of the axis of homology of the triangles ABC, XYZ is the conic which touches the sides of the former at the points A_1, B_1, C_1.

I. *Solution by* E. B. ELLIOTT, M.A.

The axis of homology of the triangles ABC, XYZ is the polar line of P with respect to the triangle ABC. Now P lies on a fixed straight line, the axis of homology of ABC, $A_1B_1C_1$. Therefore the axis of homology of

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ABC, XYZ envelopes the polar conic of the axis of homology of ABC, $A_1B_1C_1$. But (SALMON's *Higher Plane Curves*, § 185) this is the inscribed circle touching at A_1, B_1, C_1.

II. *Solution by* F. B. W. PHILLIPS.

Let $\lambda\alpha+\mu\beta+\nu\gamma=0$ be the axis of homology of ABC, $A_1B_1C_1$, and let P be (p, q, r); then the axis of homology of ABC, XYZ is

$$\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

also $$\lambda p+\mu q+\nu r=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

(2) being the tangential equation of the conic enveloped by (1). (2) written in trilinears is $\lambda^2\alpha^2+\mu^2\beta^2+\nu^2\gamma^2-2\mu\nu\beta\gamma-2\nu\lambda\gamma\alpha-2\lambda\mu\alpha\beta=0$,

which shows that the points of contact are on the lines $\lambda\alpha-\mu\beta$, $\mu\beta-\nu\gamma$, $\nu\gamma-\lambda\alpha$; hence the conic touches the sides of ABC at A_1, B_1, C_1.

4070. (Proposed by Professor CLERK-MAXWELL.)—A, B, C, D are any four points in a plane; perpendiculars are drawn from D on BC, CA, AB, and a circle is drawn through the feet of these perpendiculars; and with centre D and radius equal to the diameter of this circle another circle is drawn: if circles be similarly drawn with their centres at A, B, C, prove (1) that the four circles will intersect by threes in four points *a*, *b*, *c*, *d*. If the lines joining ABCD be a frame whose weight may be neglected, prove (2) that the stress along any line BC is measured by the corresponding line *ad* of the second figure.

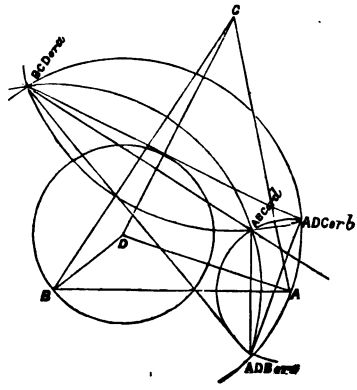
I. *Solution by* Professor WOLSTENHOLME, M.A.

For, let *abcd* be a tetragram such that any side (*bc*) is bisected at right angles by the corresponding side AD of the given tetragram, then A, B, C, D will be the centres of the circles *bcd*, *cda*, *dab*, and *abc*; let

$\angle bac=a$, $\angle cba=b$, $\angle acb=c$,

$\angle bdc=a'$, $\angle cda=b'$, $\angle adb=c'$,

and similarly for the first figure, then $a=180°-A'$, $a'=180°-A$, &c.; then, if R_1 be the radius of the circle *bcd*, R_2 of *cda*, R_3 of *dab*, R_4 of *abc*, we have



$$R_1=\frac{bc}{2\sin a'}, \quad R_4=\frac{bc}{2\sin a},$$

$$AD=R_4\cos a-R\cos a'$$
$$=\tfrac{1}{2}bc(\cot a-\cot a')$$
$$=\tfrac{1}{2}bc\frac{\sin(a'-a)}{\sin a\sin a'}=\tfrac{1}{2}bc\frac{\sin(A'-A)}{\sin A\sin A}.$$

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Therefore $R_4 = \frac{bc}{2\sin A'} = \frac{AD \sin A}{\sin(A'-A)} = \frac{BD \sin B}{\sin(B'-B)} = \frac{CD \sin C}{\sin(C'-C)}$

= radius of circle whose centre is $D = R_4$,

= diameter of circle about the feet of the perpendiculars from D on BC, CA, AB, which proves (1);

hence we have $\frac{bc}{\sin A'} = \frac{ca}{\sin B'} = \frac{ab}{\sin C'}$.

But the stresses along DA, DB, DC are as sin A', sin B', sin C'; therefore if XYZ, $x'y'z'$ be stresses along BC, CA, AB, DA, DB, DC, we have

$$\frac{X'}{bc} = \frac{Y'}{ca} = \frac{Z'}{ab},$$

and similarly each $= \frac{X}{ad} = \frac{Y}{bd} = \frac{Z}{cd}$,

or, according to Prof. Clerk-Maxwell's definition, the figure *abcd* is a force diagram reciprocal to the figure ABCD.

It follows from the above that if a conic be inscribed in ABC with D for focus, *d* is the second focus; whence the construction for a force diagram given in the Senate House Examination, Jan. 7, 1874; question 15.

4436. (Proposed by Sir James Cockle, F.R.S.)—Assuming as is proved in the *Messenger* for November, 1873 (Vol. III. pp. 108 *et seq.*), that

$$\frac{d^2u}{dx^2} + 2a\frac{du}{dx} + bu = 0 \quad \ldots\ldots (1)$$

is soluble when $\frac{d^2b}{dx^2} + 10a\frac{db}{dx} + 4b\frac{da}{dx} + 3b^2 + 24a^2b = 0 \quad \ldots\ldots (2)$;

find a modulus *c* which shall render

$$\frac{d^2u}{dx^2} + \tfrac{1}{2}\left\{\frac{1}{\sqrt{3}} - (\tan \tfrac{1}{2} am\, x)^2\right\} u = 0 \quad \ldots\ldots (3)$$

soluble by co-resolvents, the elliptic integral being of the first species.

Solution by the Proposer.

1. From (3) we have $a = 0$, and (2) becomes

$$\frac{d^2b}{dx^2} + 3b^2 = 0 \quad \ldots\ldots (4).$$

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Let $\theta = amx$ and $\left(\frac{d\theta}{dx}\right)^2 = 1 - (c \sin\theta)^2 = \Delta^2$,

then $b = \frac{1}{3}\left\{\frac{1}{\sqrt{3}} - (\tan\frac{1}{2}\theta)^2\right\}$,

and $\frac{db}{dx} = \frac{db}{d\theta}\frac{d\theta}{dx} = -\tan\frac{1}{2}\theta\,(\sec\frac{1}{2}\theta)^2\,\frac{1}{2}\Delta$.

2. Let $Q = 1 + 4(\tan\frac{1}{2}\theta)^2 + 3(\tan\frac{1}{2}\theta)^4$,

then $\frac{d^2b}{dx^2} = \frac{1}{4}\left\{2c^2\tan\frac{1}{2}\theta\,(\sec\frac{1}{2}\theta)^2 \sin\theta\cos\theta - Q\Delta^2\right\}$

$= c^2(\tan\frac{1}{2}\theta)^2\cos\theta - \frac{1}{4}Q\Delta^2$;

also $3b^2 = \frac{1}{4}\left\{Q - (4+2\sqrt{3})(\tan\frac{1}{2}\theta)^2\right\}$,

and $\frac{d^2b}{dx^2} + 3b^2 = (c^2\cos\theta - 1 - \frac{1}{3}\sqrt{3})(\tan\frac{1}{2}\theta)^2 + \frac{1}{4}Q(1-\Delta^2)$.

But $Q(1-\Delta^2) = c^2Q(\sin\theta)^2 = 4c^2Q(\cos\frac{1}{2}\theta)^4(\tan\frac{1}{2}\theta)^2$;

also $Q(\cos\frac{1}{2}\theta)^4 = 1 + 2(\sin\frac{1}{2}\theta)^2 = 2 - \cos\theta$.

Hence, substituting, $\frac{d^2b}{dx^2} + 3b^2 = \left\{2c^2 - (1+\frac{1}{2}\sqrt{3})\right\}(\tan\frac{1}{2}\theta)^2$,

and the sinister and dexter will vanish independently of θ, provided that

$$c^2 = \frac{1}{4}(2+\sqrt{3}) = \frac{1}{8}(1+\sqrt{3})^2.$$

Hence the required modulus is given by

$$c = \frac{1+\sqrt{3}}{2\sqrt{2}} = \cos 15^\circ.$$

4466. (Proposed by R. F. Scott, B.A.)—If at each point of an ellipsoid a distance k^2p^{-1} be measured along the normal, p being the perpendicular from the centre on the tangent plane, prove that the locus of the points thus obtained is another ellipsoid, the envelope of which for different values of k is the centro-surface of the original ellipsoid.

I. *Solution by the* Editor.

Let $a^2\xi^2 + b^2\nu^2 + c^2\zeta^2 - 1 = 0 \equiv U$ ..... (1)

be the tangential equation of the given ellipsoid referred to its centre and axes (see Dr. Booth's *New Geometrical Methods*, p. 62); then as $(\xi^2+\nu^2+\zeta^2)^{\frac{1}{2}}$ is the reciprocal of the perpendicular from the centre on the tangent plane, $k^2(\xi^2+\nu^2+\zeta^2)^{\frac{1}{2}}$ is the length measured along the normal; and as the normal makes with the axis of X an angle whose cosine is $p\xi$, the projection of this line on the axis of X is $k^2\xi$..... (2).

Let (x, y, z) be the projective coordinates of the point of contact of the

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tangent plane to the ellipsoid (1), and (x_1, y_1, z_1) the projective coordinates of the extremity of the normal; then $x - x_1$ is the projection of the normal on the axis of X; but this is $k^2\xi$, and as $x = a^2\xi$ (see page 62 of the same work), we have $x_1 = (a^2 - k^2)\xi$, or $\xi = \frac{x_1}{a^2 - k^2}$ ..... (3).

In like manner we find $v = \frac{y_1}{b^2 - k^2}$, and $\zeta = \frac{z_1}{c^2 - k^2}$.

Substituting these values of ξ, v, ζ in the tangential equation (1) of the ellipsoid, we shall have for the locus of the extremity of the normal, the ellipsoid whose projective equation is

$$\frac{a^2x_1^2}{(a^2-k^2)^2} + \frac{b^2y_1^2}{(b^2-k^2)^2} + \frac{c^2z_1^2}{(c^2-k^2)^2} - 1 = 0 \equiv W \text{ ..... (4)},$$

and therefore the tangential equation of this ellipsoid will be

$$(a^2-k^2)^2\frac{\xi^2}{a^2} + (b^2-k^2)^2\frac{v^2}{b^2} + (c^2-k^2)^2\frac{\zeta^2}{c^2} - 1 \equiv V = 0 \text{ ..... (5)}.$$

Now to determine the envelope of this surface we must eliminate k between $V=0$, and $\frac{dV}{dk} = 0$. But $\frac{dV}{dk} = 0$ gives

$$\xi^2 + v^2 + \zeta^2 = k^2\left(\frac{\xi^2}{a^2} + \frac{v^2}{b^2} + \frac{\zeta^2}{c^2}\right) \text{ ..... (6)}.$$

Substituting this value of k in the preceding equation, we shall have finally for the tangential equation of the envelope

$$(\xi^2 + v^2 + \zeta^2)^2 = \left(\frac{\xi^2}{a^2} + \frac{v^2}{b^2} + \frac{\zeta^2}{c^2}\right)(a^2\xi^2 + b^2v^2 + c^2\zeta^2 - 1).$$

But this is the tangential equation of the *Surface of Centres of Curvature of the Ellipsoid* (1), or as it is called by the Proposer of the question, *the Centro-surface*, as shown in Dr. Booth's *New Geometrical Methods*, p. 113.

II. *Solution by* E. B. Elliott, B.A.

Let (x, y, z) be a point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ ..... (1)},$$

and (ξ, η, ζ) the corresponding point on the required locus; then we have

$$\frac{\xi - x}{\frac{x}{a^2}} = \frac{\eta - y}{\frac{y}{b^2}} = \frac{\zeta - z}{\frac{z}{c^2}} = \left\{\frac{(\xi-x)^2 + (\eta-y)^2 + (\zeta-z)^2}{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}\right\}^{\frac{1}{2}} = \frac{k\,p^{-1}}{p^{-1}} = k^2;$$

therefore $\xi = x\left(1 + \frac{k^2}{a^2}\right)$, $x = \frac{a^2\xi}{a^2 + k^2}$, $y = \frac{b^2\eta}{b^2 + k^2}$, $z = \frac{c^2\zeta}{c^2 + k^2}$.

Substituting these values in (1), we have

$$\frac{a^2\xi^2}{(a^2+k^2)^2} + \frac{b^2\eta^2}{(b^2+k^2)^2} + \frac{c^2\zeta^2}{(c^2+k^2)^2} = 1 \text{ ..... (2)},$$

and the locus is an ellipsoid.

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The envelope is found by eliminating k^2 between (2) and its differential with regard to k^2; or, which is the same thing, by equating to zero the discriminant of (2) written as a quantic in k^2. But this is known to give the equation of the surface of centres of (1).

III. *Solution by* R. Tucker, M.A.

Taking the ellipsoid $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the equation to a normal is

$$\frac{x-X}{a^{-2}x}=\frac{y-Y}{b^{-2}y}=\frac{z-Z}{c^{-2}z}, \text{ which } =\frac{k^2p^{-1}}{p^{-1}}=k^2,$$

where X Y Z is the point whose locus is required; hence $X=a^{-2}x\,(a^2-k^2)$. Therefore the locus required is

$$\frac{a^2}{(a^2-k^2)^2}X^2+\frac{b^2}{(b^2-k^2)^2}Y^2+\frac{c^2}{(c^2-k^2)^2}Z^2=1 \quad \ldots\ldots\ldots (2);$$

and this, taken in connexion with its derived equation

$$\frac{a^2X^2}{(a^2-k^2)^3}+\frac{b^2Y^2}{(b^2-k^2)^3}+\frac{c^2Z^2}{(c^2-k^2)^3}=0 \quad \ldots\ldots\ldots\ldots\ldots (3),$$

will give the envelope of the second ellipsoid.

But the elimination of k^2 between (2) and (3) gives the centro-surface of the original ellipsoid (Salmon's *Geometry of Three Dimensions*, § 502, 3rd edition).

4376. (Proposed by W. Niven, M.A.)—A uniform circular wire of radius a, charged with electricity of line-density e, surrounds an uninsulated concentric spherical conductor of radius c; prove that the electrical density at any point of the surface of the conductor is

$$-\frac{e}{2c}\left(1-5\cdot\frac{1}{2}\cdot\frac{c^2}{a^2}Q_2+9\cdot\frac{1\cdot3}{2\cdot4}\frac{c^4}{a^4}Q_4-13\cdot\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{c^6}{a^6}Q_6+\ldots\ldots\right);$$

where $Q_1, Q_2, Q_3, \ldots\ldots$ denote the zonal harmonics at the point, the pole of the plane of the wire being the pole of the harmonics.

Solution by H. Hilary, B.A.

The electric image of the given ring will be a concentric ring of radius a', where $aa'=c^2$.

Let ρ be the line-density of the image. For points external to the sphere $r=c$, the potential of the ring a' will be the same as that of the electricity induced on the uninsulated conductor by the ring a. We will

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find the potential of the ring a'. For any point in the axis, distant r from the centre, the potential

$$= \frac{2\pi a'\rho}{(a'^2+r^2)^{\frac{1}{2}}} = \frac{2\pi a'\rho}{r}\left(1 - \frac{1}{2}\frac{a'^2}{r^2} + \frac{1.3}{2.4}\frac{a'^4}{r^4} - \ldots\ldots\right), \quad r > a'.$$

Consider now the series (see Thomson and Taits' *Nat. Phil.*, p. 406)

$$v \equiv 2\pi\rho\left(\frac{a'}{r} - \frac{1}{2}\cdot\frac{a'^3}{r^3}Q_2 + \frac{1.3}{2.4}\frac{a'^5}{r^5}Q_4 - \ldots\ldots\right), \quad r > a'.$$

Since $\frac{a^{i+1}Q_i}{r^{i+1}}$ is a complete solid harmonic of negative degree, v is a potential function and satisfies Laplace's equation

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0;$$

therefore so also does $V - v$, where V is the potential of the image for the point (r, θ).

Now for points on the axis, since there $Q_1, Q_2, \ldots\ldots$ are all unity, V and v are identical; that is, for a finite portion of space, $V - v \equiv U$ vanishes. Therefore by a well known theorem of Gauss, $U = 0$ over all the region which can be reached in any way without passing through electrified matter. Hence for all points P ($r > a'$), the series v is the potential of the electric image.

Call, for a moment, this potential $\phi(r)$ for points external to the sphere $r = c$. $\phi(r)$ is the potential of the charge induced on the sphere by the influence of the ring a; therefore the potential of the charge for points internal to the sphere

$$\equiv v' = \frac{c}{r}\phi\left(\frac{c^2}{r}\right) = 2\pi\rho\left(\frac{a'}{c} - \frac{1}{2}\frac{a'^3r^2}{c^5}Q_2 + \frac{1.3}{2.4}\frac{a'^5r^4}{c^9}Q_4 - \ldots\ldots\right),$$

where $r > a'$, but $< c$.

But if σ be the surface-density of the induced electricity

$$\left(\frac{dv}{dr}\right)_{r=c} - \left(\frac{dv'}{dr}\right)_{r=c} = -4\pi\sigma.$$

Now $$\left(\frac{dv}{dr}\right)_{r=c} = -2\pi\rho\left(\frac{a'}{c^2} - 3\cdot\frac{1}{2}\frac{a'^3}{c^4}Q_2 + 5\cdot\frac{1.3}{2.4}\frac{a'^5}{c^6}Q_4 - \ldots\ldots\right),$$

$$\left(\frac{dv'}{dr}\right)_{r=c} = 2\pi\rho\left(-2\cdot\frac{1}{2}\cdot\frac{a'^3}{c^4}Q_2 + 4\cdot\frac{1.3}{2.4}\frac{a'^5}{c^6}Q_4 - \ldots\ldots\right);$$

therefore $$-4\pi\sigma = -2\pi\rho\left(\frac{a'}{c^2} - 5\cdot\frac{1}{2}\frac{a'^3}{c^4}Q_2 + 9\cdot\frac{1.3}{2.4}\frac{a'^5}{c^6}Q_4 - \ldots\ldots\right)$$

$$= -2\pi\rho\left(\frac{1}{a} - 5\cdot\frac{1}{2}\frac{c^2}{a^3}Q_2 + 9\cdot\frac{1.3}{2.4}\frac{c^4}{a^5}Q_4 - \ldots\ldots\right).$$

To find ρ, let ds and ds' be corresponding elements of the rings; then

$$\rho ds' = -\frac{c}{a}e ds, \text{ but } \frac{ds}{ds'} = \frac{a}{a'} = \frac{a^2}{c^2}; \text{ therefore } \rho = -\frac{ae}{c};$$

therefore $$\sigma = -\frac{e}{2c}\left(1 - 5\cdot\frac{1}{2}\cdot\frac{c^2}{a^2}Q_2 + 9\cdot\frac{1.3}{2.4}\frac{c^4}{a^4}Q_4 - \ldots\ldots\right).$$

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The total *quantity* of electricity on the uninsulated sphere

$$= \iint \sigma dS = -\frac{e}{2c} \iint dS = -2\pi ec$$

$$= -\frac{c}{a} \cdot 2\pi as = -\frac{c}{a} \times \text{(quantity in the inducing mass)};$$

as was to be expected.

4460. (Proposed by Professor CROFTON, F.R.S.)—Show that a perfectly flexible, heavy, uniform string may be made to rest (in unstable equilibrium) in the form of an arch. Find, also, the form of an arch composed of a number of smooth equal spheres, and the curve formed by the arch when the number of spheres is continually increased.

Solution by Professor TOWNSEND, F.R.S.

Denoting, as regards the second part, by ABCD...MN the polygon determined by the centres of the several spheres, taken from either terminal in consecutive order; then since, for equilibrium, the sides AB and CD must intersect on the vertical through the middle point of BC, the sides BC and DE on that through the middle point of CD, the sides CD and EF on that through the middle point of DE, &c., the polygon of equilibrium ABCD...MN is therefore the same as of a framework of uniform and equal bars AB, BC, CD, &c., starting without friction from the two terminals A and N, and capable of free motion round the several hinges B, C, D, &c.; and becomes, consequently, the common catenary or chainette, reversed, when the number of its sides is indefinitely increased, and their common length indefinitely diminished. As regards the first part, if a flexible cord, whether of uniform or of variable thickness, were incompressible as well as inextensible, like the framework above, the reflexion, with respect to the horizontal line passing through its two terminal supports, of its curve of stable equilibrium under the action of gravity would manifestly be a form in which, if it were adjusted, it would be in unstable equilibrium under the same action; this of course would not be the case if, on the ordinary hypothesis, like a chain composed of ring links, it were only inextensible.

4275. (Proposed by W. H. H. HUDSON, M.A.)—If S, H be the poles of a Lemniscate, P any point on it, C the middle point of SH, and if PH′, PS′ be taken on PS, PH, equal respectively to PH, PS; prove that the normal at P bisects S′H′.

Solution by Professor ALLMAN, LL.D.

This question is a particular case of one for giving a geometrical construction for the determination of the normal at a point of the surface

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$F(\rho_1, \rho_2, \dots \rho_n) = \text{const.}$, where $\rho_1, \rho_2, \dots \rho_n$ are the distances from any point on the surface to n fixed points, the solution of which is as follows:—

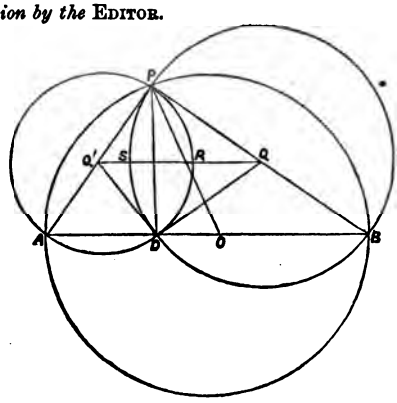
Connect the given point on the surface with the n fixed points, and on these lines measure lengths proportional to $\frac{dF}{d\rho_1}, \frac{dF}{d\rho_2}, \dots \frac{dF}{d\rho_n}$ respectively; the direction of the normal at the point is the *resultant* of the lines thus determined.

Applying this method to the curve $\rho_1\rho_2 = \text{const.}$, we get at once the construction in Quest. 4275. Several examples in Williamson's *Differential Calculus*, Chap. xii., are solved immediately by this method. The principle on which this method is founded was first given by Leibnitz.

[Other Solutions are given on p. 46 of Vol. XXI. of the *Reprint*.]

4360. (Proposed by A. Martin.)—From the ends of a diameter of a given circle lines are drawn to a point in the circumference, and on these as diameters circles are drawn; prove that the mean area common to these two circles, in parts of the area of the given circle, is $\frac{1}{4} - \frac{2}{\pi^2}$.

Solution by the Editor.



Let O be the centre of the given circle; P the point on its circumference; Q and Q' the centres of the circles on BP and PA; and PRDS the common area whose average is required. Then, taking the radius of the given circle as unity, and putting $\angle OAP = \theta$, we have $\angle PQ'D = 2\theta$ and $Q'P = \cos\theta$, therefore

sector $PRDQ' = \theta\cos^2\theta$;

and similarly we have

sector $PSDQ = (\frac{1}{2}\pi - \theta)\sin^2\theta$;

also Area of quadrilateral $PQDQ' = \frac{1}{2}\triangle ABP = \frac{1}{2}\sin 2\theta$;

hence we have

Area PRDS = ~~sector~~ PRDQ' + sector PSDQ − quad. PQDQ'

$= \frac{1}{4}\pi - \frac{1}{2}\sin 2\theta - (\frac{1}{4}\pi - \theta)\cos 2\theta$;

hence putting $2\theta = \phi = \angle POB$, the average value of this area is

$$\frac{1}{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\left\{\tfrac{1}{4}\pi - \tfrac{1}{2}\sin\phi - \tfrac{1}{2}(\tfrac{1}{2}\pi - \phi)\cos\phi\right\}d\phi = \frac{2}{\pi}\left\{\frac{\pi^2}{8} - \frac{1}{2} - \frac{1}{2}\right\}$$

$$= \frac{\pi}{4} - \frac{2}{\pi} = \left(\frac{1}{4} - \frac{2}{\pi^2}\right) \text{ of area of given circle.}$$

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4462. (Proposed by Rev. Dr. Booth, F.R.S.)—In Maclaurin's "*Tractatus de Linearum Geometricarum proprietatibus generalibus*," p. 11, it is shown by Algebra that in any curve the sum of the reciprocals of the subtangents made by a vector revolving round a given point is a constant. Give a geometrical proof of this theorem in the case of the conic sections.

Solution by R. Tucker, M.A.

Let OAB be the fixed line, and OTT' any variable secant; draw tangents at T, T' meeting in P and let the tangents at A, B meet in Q, then PQ is the polar of O.

Hence (Besant's *Conics*, § 190)

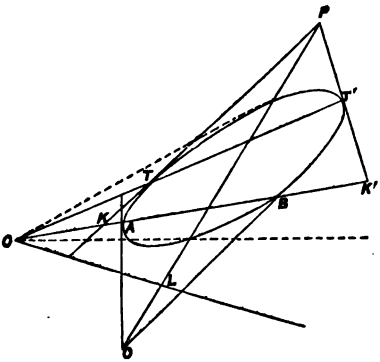
$$\frac{OK^2}{OK'^2} = \frac{KA \cdot KB}{K'A \cdot K'B},$$

and we obtain

$$\frac{1}{OK} + \frac{1}{OK'} = \frac{1}{OA} + \frac{1}{OB}.$$

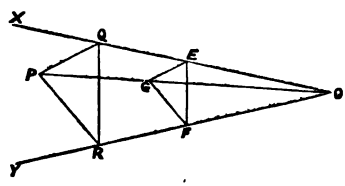
If AOB (as Maclaurin specifies) does not meet the conic, but cuts the polar in L, then since P (OKLK') is an harmonic pencil, we should have

$$\frac{1}{OK} + \frac{1}{OK'} = \frac{2}{OL}.$$

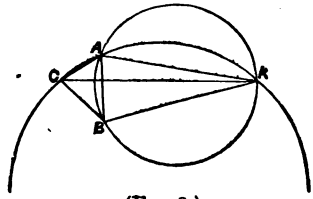


4149. (Proposed by T. T. Wilkinson, F.R.A.S.)—Construct a triangle similar to a given triangle, such that two of its angles may rest on lines given in position, and its third angle touch a given point.

I. *Solution by* A. Renshaw; Belle Easton; *and others.*



(Fig. 1.)



(Fig. 2.)

Let OX, OY (Fig. 1) be the lines that two of the angles are to rest upon, and P the given point to be touched by the third angle of the triangle to be constructed; and let ABC (Fig. 2) be that to which it is to be similar. On AB construct the segment AKB containing the angle

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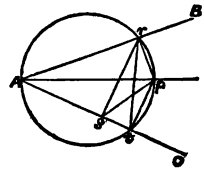
AKB = XOY, and on AC the segment AKC containing the angle AKC = XOP. Join A, B, C with the point K, where the circles cut one another; make OE = KA, and the figure OEGF equal in all respects to KACB. Draw PQ parallel to EG, and PR parallel to GF, and join QR, which will consequently be parallel to EF. Then PQR will be equiangular to ABC, and its angles will be placed as required.

[Mr. MURPHY remarks that as one angle of a triangle given in species turns round a given point, and another angle moves along a given straight line, the locus of the third angle is, by a well known property, a straight line, the intersection of which with the second straight line gives the second vertex, and hence, as the species of the triangle is given, it can be constructed. He adds that the question might be more generally proposed as follows:—One angle turns round a point, another angle moves on a straight line or circle, the third on a circle, or any other curve; construct the triangle. The solution is effected by the intersection of the locus of the third angle with a given curve.]

II. *Solution by the* PROPOSER.



Analysis. Suppose the problem solved:—AB, AC the lines given in position; P the given point; and P*rs* the triangle similar to the given one. Let a circle be drawn through the points A, *r*, P, cutting AC in *t*; and join P*t*, *rt*. Then $\angle$ P*rt* = PA*t* = a given angle; the $\angle$ P*tr* = PA*r* is also given; and $\angle$ *r*P*t* = supplement to BAC, and is also given.

The sides *rt*, *rs*, and the $\angle$ *trs* are therefore given; and hence the $\angle$ *rts* is a given angle. The quadrilateral *rst*P is therefore given in species; and consequently the $\angle$ *rts* = *r*PA is also given; and the following construction is obvious. Join PA; make the $\angle$ AP*r* = the given $\angle$ A*tr*; also draw *rs*, making with P*r* the given angle, and join P*s*. Then P*rs* is the triangle required.

4301. (Proposed by E. B. ELLIOTT.)—Two points are taken on an ellipse, all values of their eccentric angles being equally likely, and the tangents at them are drawn. Find the chance that these intersect in the area contained between two given ellipses, similar, similarly situated, and concentric with the first.

Solution by A. M. NASH; *the* PROPOSER; *and others.*

The three ellipses being respectively

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\alpha, \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\beta;$$

let θ, ϕ be the eccentric angles of any two points on the first; then the coordinates of the point of intersection of tangents at them are

$$x = a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \quad y = b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)};$$

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therefore this intersection lies on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 \tfrac{1}{2}(\theta - \phi).$$

Hence in favourable cases $\frac{1}{2}(\theta - \phi) > \alpha < \beta$, so that $\phi < \theta - 2\alpha > \theta - 2\beta$; and thus the probability required is

$$\left\{ \int_0^{2\pi} \int_{\theta-2\beta}^{\theta-2\alpha} d\theta\, d\phi \right\} \div \int_0^{2\pi} \int_0^{\theta} d\theta\, d\phi = \frac{2(\beta - \alpha)}{\pi}.$$

4456. (From WHITWORTH's *Choice and Chance.*)—If three numbers be named at random, prove that they are just as likely as not to be proportional to the sides of a possible triangle.

I. *Solution by* S. FORDE; C. LEUDESDORF; *and others.*

Let a, b, c denote the three numbers, and let them be supposed to lie between 0 and n. Assuming them to be all unequal, let $a > b > c$. If a, b, c are proportional to the sides of a triangle, $a > b + c$ and two similar inequalities, which will be satisfied, because $a > b > c$. Also the chance that a, b, c are all unequal is

$$1 - \frac{3}{n^2}\left(1 - \frac{1}{n}\right) - \frac{1}{n^3} \quad \text{or} \quad \left(1 - \frac{1}{n}\right)^2 \left(\frac{2}{n} + 1\right).$$

Therefore the chance required is

$$\left(1 - \frac{1}{n}\right)^2 \left(\frac{2}{n} + 1\right) \int_0^n dc \int_c^n db \int_0^{b+c} da \div \int_0^n \int_0^n \int_0^n da\, db\, dc;$$

or

$$\frac{1}{2}\left(1 - \frac{1}{n}\right)^2 \left(\frac{2}{n} + 1\right).$$

Let now n become infinite; then the chance becomes $\frac{1}{2}$.

II. *Solution by* H. S. MONCK.

The order of drawing the three numbers being immaterial, let us suppose the largest number taken first (it will probably be very large, since we have to select from all numbers from 1 up to ∞). The question then is reduced to this, to show that in taking two numbers b and c both less than the given one (a), it is as likely that $b + c$ will be greater than a as that it will be less. First then the chances are equal that b will be less than $\frac{1}{2}a$ and greater than $\frac{1}{2}a$, and in either event a similar chance arises as to c. There are thus four equally probable cases,

$b < \frac{1}{2}a$, $c < \frac{1}{2}a$; $b < \frac{1}{2}a$, $c > \frac{1}{2}a$; $b > \frac{1}{2}a$, $c < \frac{1}{2}a$; and $b > \frac{1}{2}a$, $c > \frac{1}{2}a$.

In the first of these the three numbers are not proportional to the sides of a possible triangle; in the fourth they are so, and the second and third are still indeterminate. It is evident, however, that in the case of $b < \frac{1}{2}a$, $c > \frac{1}{2}a$, it is just as likely that c will exceed $\frac{1}{2}a$ by a greater amount than b falls short of it, as that it will do so by a less amount, and similarly for the

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case of $b > \frac{1}{2}a$, $c < \frac{1}{2}a$. Whence the entire chance is equal. This however is only accurate where a is a very large number; in fact, strictly for $a = \infty$. If $a = 5$, for example, the chance against b and c forming the other two sides of a triangle is 2 to 1. This arises from the chance being adverse if $b + c = a$.

4479. (Proposed by CHRISTINE LADD.)—The radii of the fore and hind wheels of a coach are r and R, and a is the distance between their centres. A particle driven from the highest point of the hind wheel falls on the highest point of the fore wheel; find the velocity of the coach.

Solution by G. S. CARR.

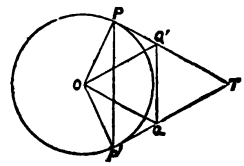
Let v be the velocity of the coach, therefore $2v$ will be the actual velocity of the particle, and v its relative velocity. It has to fall from rest vertically under the action of gravity through the space $2(R-r)$, while it advances through the horizontal distance $\{a^2-(R-r)^2\}^{\frac{1}{2}}$ with the relative velocity u. Hence

$$2(R-r) = \tfrac{1}{2}gt^2, \text{ and } u^2t^2 = a^2-(R-r)^2.$$

Eliminating t, we have $$u^2 = \frac{a^2-(R-r)^2}{4(R-r)}\,g.$$

4443. (Proposed by J. J. WALKER, M.A.)—Tangents are drawn to an ellipse at P, P′, and from the centre O, lines OQ, OQ′ parallel to either tangent, and meeting the other in Q, Q′ respectively; prove that the triangle contained by the semi-axes is a mean proportional between the triangles POP′, QOQ′.

I. *Solution by* A. T. P. SHEPHERD; E. B. ELLIOTT, B.A.; *and others.*



Project the figure orthogonally, so that the ellipse may become the circle in the figure; and let $PO = a$, and $\angle PTP' = \alpha$; then

$$QO = a \operatorname{cosec} \alpha = Q'O;$$

hence we have

area $\triangle QOQ' = \frac{1}{2}QO \cdot Q'O \sin\alpha = \frac{1}{2}a^2 \operatorname{cosec}\alpha$,

and area $\triangle POP' = \frac{1}{2}a^2 \sin POP' = \frac{1}{2}a^2 \sin\alpha$;

therefore $\triangle POP' \cdot \triangle QOQ' = \frac{1}{4}a^4$.

Hence the triangle contained by two radii at right angles is a mean proportional between the triangles POP′, QOQ′; or, in the unprojected figure, if b be the major axis, we have,

$$\triangle POP' \cdot \triangle QOQ' = \frac{b^2}{a^2} \cdot \frac{a^4}{4} = (\tfrac{1}{2}ab)^2.$$

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II. *Solution by* R. TUCKER, M.A.; S. FORDE; *and others.*

The tangents at the points P, P' (ϕ, ϕ') are

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1, \quad \frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = 1 \ldots\ldots (1,2);$$

hence the coordinates of Q' and Q are respectively

$$\frac{a\sin\phi'}{\sin(\phi'-\phi)}, \ -\frac{b\cos\phi'}{\sin(\phi'-\phi)}; \quad -\frac{a\sin\phi}{\sin(\phi'-\phi)}, \ \frac{b\cos\phi}{\sin(\phi'-\phi)}.$$

Hence $\quad \Delta QOQ' = \frac{1}{2}\left(\frac{ab}{\sin(\phi'-\phi)}\right)$, and $\Delta POP' = \frac{1}{2}ab\sin(\phi'-\phi)$,

whence the truth of the proposition is manifest.

4176. (Proposed by J. C. MALET, M.A.)—Using the ordinary notation of "Elliptic Functions," prove the following formula for an Elliptic Function of the first kind with an imaginary modulus $k+ik'$, where k is less than unity, $i = \sqrt{(-1)}$, and $k' = (1-k^2)^{\frac{1}{2}}$, viz.

$$F\{k+ik', \phi\} = AF(\lambda, \psi) + BF(\mu, \theta);$$

A, B, λ, μ being real quantities, λ and μ less than unity; find their values and also the values of the amplitudes ψ and θ.

Solution by the PROPOSER.

By a transformation analogous to Gauss's, we have

$$F(c, \phi) = \frac{1}{2c^{\frac{1}{2}}} F\left(\frac{1+c}{2\sqrt{c}}, \psi\right), \quad \text{where} \quad \sin\psi = \frac{2c^{\frac{1}{2}}\sin\phi}{1+c\sin^2\phi}.$$

If now $c = k+ik'$, we have

$$\frac{(1+c)^2}{4c} = \frac{1+k}{2}; \ \text{therefore} \ 2c^{\frac{1}{2}} = \left(\frac{2}{1+k}\right)^{\frac{1}{2}}(1+k+ik').$$

Hence we have

$$F(k+ik', \phi) = \left(\frac{1+k}{2}\right)^{\frac{1}{2}} \frac{1}{1+k+ik'} F\left\{\left(\frac{1+k}{2}\right)^{\frac{1}{2}}, \psi\right\}$$

$$= \frac{(1+k)^{\frac{1}{2}}}{(\sqrt{2})^3} F\left\{\left(\frac{1+k}{2}\right)^{\frac{1}{2}}, \psi\right\} - \frac{ik'}{\{2^2(1+k)\}^{\frac{1}{2}}} F\left\{\left(\frac{1+k}{2}\right)^{\frac{1}{2}}, \psi\right\}$$

In the second part of the last expression, if we substitute for $\sin\psi$, $i\tan\theta$, we have $\quad F\left\{\left(\frac{1+k}{2}\right)^{\frac{1}{2}}, \psi\right\} = iF\left\{\left(\frac{1-k}{2}\right)^{\frac{1}{2}}, \theta\right\}$.

Hence we have finally

$$F(k+ik', \phi) = \left(\frac{1+k}{8}\right)^{\frac{1}{2}} F\left\{\left(\frac{1+k}{2}\right)^{\frac{1}{2}}, \psi\right\} + \left(\frac{1-k}{8}\right)^{\frac{1}{2}} F\left\{\left(\frac{1-k}{2}\right)^{\frac{1}{2}}, \theta\right\}.$$

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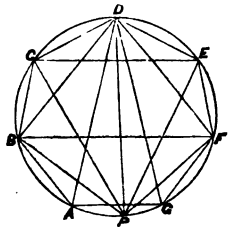
where
$$\sin\psi = \left(\frac{2}{1+k}\right)^{\frac{1}{2}} \frac{(1+k+ik')\sin\phi}{1+(1+k+ik')\sin^2\phi},$$
$$\tan\theta = \left(\frac{2}{1+k}\right)^{\frac{1}{2}} \frac{\{k'-i(1+k)\}\sin\phi}{1+(1+k+ik')\sin^2\phi}.$$

4269. (Proposed by the Rev. Dr. Booth, F.R.S.)—If a, b, c, d, e, f, g be chords drawn from any point on the circumference of a circle to the seven angles of an inscribed regular heptagon; prove that
$$(a+g)(b+f)(c+e) = d^3, \quad a+c+e+g = b+d+f \ldots\ldots (1, 2).$$

I. *Solution by* A. B. Evans, M.A.; M. Collins, LL.D.; *and others.*



Let P be any point on the circumference, and let PA, PB, PC, PD, PE, PF, PG be the seven chords represented by a, b, c, d, e, f, g respectively. Let m be a side of the regular heptagon, and put
$$BD = DF = CE = n,$$
$$AD = GD = BF = p.$$

From the three inscribed quadrilaterals PADG, PBDF, PCDE, we have
$$\left.\begin{aligned} PD.AG &= PA.DG + PG.AD \\ PD.BF &= PB.DF + PF.BD \\ PD.CE &= PC.DE + PE.CD \end{aligned}\right\}, \text{ or } \left.\begin{aligned} dm &= (a+g)p \\ dp &= (b+f)n \\ dn &= (c+e)m \end{aligned}\right\} \ldots\ldots (3).$$

The product of equations (3) gives equation (1). Again, from the four inscribed quadrilaterals PGFE, ABCP, PFED, PBDE, we have
$$\left.\begin{aligned} PE.GF + PG.EF &= PF.GE, \text{ or } (e+g)m = fn \\ AP.BC + AB.CP &= AC.BP, \text{ or } (a+c)m = bn \end{aligned}\right\} \ldots\ldots (4),$$
$$\left.\begin{aligned} PD.EF + PF.DE &= PE.DF, \text{ or } (d+f)m = en \\ BP.DE + BD.PE &= PD.BE, \text{ or } bm + en = dp \end{aligned}\right\} \ldots\ldots (5).$$

From (4), $$(a+c+e+g)m = (b+f)n,$$

and from (5) and (3), $$(b+d+f)m = dp = (b+f)n;$$

whence $$a+c+e+g = b+d+f.$$

II. *Solution by* S. Forde; C. Leudesdorf; *and others.*

If the chord a make an angle θ with the diameter (of length unity) passing through the fixed point, then
$$a = \cos\theta, \quad b = \cos(\theta+\phi), \quad c = \cos(\theta+2\phi), \ldots\ldots, \text{ where } 7\phi = \pi;$$

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therefore $\dfrac{(a+g)(b+f)(c+e)}{d^3}$

$$= \frac{[\cos\theta+\cos(\theta+6\phi)][\cos(\theta+\phi)+\cos(\theta+5\phi)][\cos(\theta+2\phi)+\cos(\theta+4\phi)]}{\cos^3(\theta+3\phi)}$$

$$= 8\cos 3\phi\cos 2\phi\cos\phi = \frac{2\cos 3\phi\sin 4\phi}{\sin\phi} = \frac{\sin 7\phi}{\sin\phi} + 1 = 1 \quad \ldots\ldots\ldots (1);$$

$$a+c+e+g-(b+d+f) = \sum_{r=6}^{r=0}\cos[\theta+r(\pi+\phi)]$$

$$= \cos[\theta+3(\pi+\phi)]\frac{\sin\frac{7}{2}(\pi+\phi)}{\sin\frac{1}{2}(\pi+\phi)} = 0 \quad \ldots\ldots (2).$$

4406. (Proposed by J. L. McKenzie.)—Find the least triangle whose sides are integers in arithmetical progression, and the perpendiculars on the sides from the opposite angles integers in harmonical progression.

Solution by Asher B. Evans, M.A.

Let a, b, c be the sides of the triangle; p_1, p_2, p_3 the perpendiculars; and r the radius of the inscribed circle; then the following are general expressions for the sides and perpendiculars:

$$a = r(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C), \quad b = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}C), \quad c = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B),$$

$$p_1 = \frac{r(a+b+c)}{a}, \quad p_2 = \frac{r(a+b+c)}{b}, \quad p_3 = \frac{r(a+b+c)}{c} \ldots\ldots (1).$$

As p_1, p_2, p_3 vary reciprocally as a, b, c, the former are in harmonical progression when the latter are in arithmetical. The condition $a+c=2b$ gives

$$2\cot\tfrac{1}{2}B = \cot\tfrac{1}{2}A + \cot\tfrac{1}{2}C \ldots\ldots\ldots (2).$$

Put $\cot\frac{1}{2}A = x$ and $\cot\frac{1}{2}B = y$, then $\cot\frac{1}{2}C = \tan\dfrac{A+B}{2} = \dfrac{x+y}{xy-1}$;

and from (2), $2y = x + \dfrac{x+y}{xy-1}$, therefore $y = \dfrac{x^2+3}{2x}$ and $\dfrac{x+y}{xy-1} = \dfrac{3}{x}$.

Hence $$\cot\tfrac{1}{2}A = x, \quad \cot\tfrac{1}{2}B = \frac{x^2+3}{2x}, \quad \cot\tfrac{1}{2}C = \frac{3}{x} \ldots\ldots\ldots (3).$$

From (1) and (3) we find

$$\left.\begin{aligned} a &= \left(\frac{x^2+9}{2x}\right)r, \quad b = \left(\frac{x^2+3}{x}\right)r, \quad c = \left(\frac{3x^2+3}{2x}\right)r, \\ p_1 &= \left(\frac{6x^2+18}{x^2+9}\right)r, \quad p_2 = 3r, \quad p_3 = \left(\frac{2x^2+6}{x^2+1}\right)r, \end{aligned}\right\} \ldots\ldots (4).$$

Equation (4) contains all the solutions of the question; but on giving any value to x, care must be taken to give r such a value as shall make a, b, c, p_1, p_2, p_3 integral. When $x=1$ and $r=5$, we have $a=25$, $b=20$, $c=15$, $p_1=12$, $p_2=15$, $p_3=20$; which are evidently the smallest integral numbers that will satisfy the required conditions.

[Another Solution is given on p. 108 of Vol. XXI. of the *Reprint*.]

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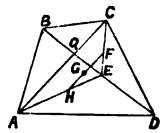
4297. (Proposed by J. J. WALKER, M.A.)—If h_1, h_2, h_3, h_4, h_0 are the distances of the four corners and the intersection of diagonals respectively of a plane quadrilateral from any other plane, prove that the distance of the centre of gravity of its area is equal to

$$\tfrac{1}{3}(h_1+h_2+h_3+h_4-h_0).$$

Solution by ASHER B. EVANS, M.A.

Let F_1, H_1, G_1 be the distances of the centres of gravity F, H, G of BCD, BAD, ABCD from the plane. It is evident that CF and AH produced intersect in E the middle point of BD, and that CF = 2FE, AH = 2HE, and



$$\frac{HG}{FG}=\frac{\Delta BCD}{\Delta BAD}=\frac{CQ}{AQ} \quad \ldots\ldots\ldots\ldots (1).$$

Also $E_1=\frac{1}{2}(h_2+h_4)$, $F_1=\frac{1}{3}(h_2+h_3+h_4)$, $H_1=\frac{1}{3}(h_1+h_2+h_4)$,

$$G_1=\frac{HG}{HF}\cdot F_1+\frac{FG}{HF}\cdot H_1, \quad h_0=\frac{AQ}{AC}\cdot h_2+\frac{CQ}{AC}\cdot h_1;$$

whence $$\frac{AQ}{AC}=\frac{h_0-h_1}{h_3-h_1}, \quad \frac{CQ}{AC}=\frac{h_3-h_0}{h_3-h_1};$$

and by the aid of (1)

$$\frac{HG}{FG}=\frac{h_3-h_0}{h_0-h_1}, \quad \frac{HG}{HF}=\frac{h_3-h_0}{h_3-h_1}, \quad \frac{FG}{HF}=\frac{h_0-h_1}{h_3-h_1};$$

therefore $$3G_1=\left(\frac{h_3-h_0}{h_3-h_1}\right)(h_2+h_3-h_4)+\left(\frac{h_0-h_1}{h_3-h_1}\right)(h_1+h_2+h_4),$$

and $$G_1=\tfrac{1}{3}(h_1+h_2+h_3+h_4-h_0).$$

4257. (Proposed by A. MARTIN.)—A hole $2a$ inches square, is made through the centre of a sphere of radius r; find (1) the surface, (2) the volume removed.

Solution by the PROPOSER.

Taking the origin at the centre of the sphere, its equation is

$$x^2+y^2+z^2=r^2.$$

Putting S for the whole surface removed, we have

$$S=2\iint\left(1+\frac{dz^2}{dx^2}+\frac{dz^2}{dy^2}\right)^{\frac{1}{2}}dx\,dy.$$

But $$\frac{dz}{dx}=-\frac{x}{z}, \text{ and } \frac{dz}{dy}=-\frac{y}{z};$$

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therefore $S = 2\int_{-a}^{+a}\int_{-a}^{+a}\frac{r\,dx\,dy}{z}$

$$= 2\int_{-a}^{+a}\int_{-a}^{+a}\frac{r\,dx\,dy}{(r^2-x^2-y^2)^{\frac{1}{2}}} = 4r\int_{-a}^{+a}\sin^{-1}\left(\frac{a}{(r^2-x^2)^{\frac{1}{2}}}\right)dx$$

$$= 16ar\sin^{-1}\left(\frac{a}{(r^2-a^2)^{\frac{1}{2}}}\right) - 8r^2\sin^{-1}\left(\frac{a^2}{r^2-a^2}\right).$$

Let V be the whole volume removed, then we have

$$V = 2\iiint dx\,dy\,dz = 2\iint z\,dx\,dy$$

$$= 2\int_{-a}^{+a}\int_{-a}^{+a}(r^2-x^2-y^2)^{\frac{1}{2}}\,dx\,dy$$

$$= 2a\int_{-a}^{+a}(r^2-a^2-x^2)^{\frac{1}{2}}\,dx + 2\int_{-a}^{+a}(r^2-x^2)\sin^{-1}\left(\frac{a}{(r^2-x^2)^{\frac{1}{2}}}\right)dx$$

$$= \tfrac{2}{3}a^2(r^2-2a^2)^{\frac{1}{2}} + \tfrac{2}{3}a(3r^2-a^2)\sin^{-1}\left(\frac{a}{(r^2-a^2)^{\frac{1}{2}}}\right) - \tfrac{2}{3}r^3\sin^{-1}\left(\frac{a^2}{r^2-a^2}\right).$$

4477. (Proposed by W. H. H. Hudson, M.A.)—Prove that the expression $(xyz+x^2y-y^2z+z^2x)^2+(xyz+xy^2+yz^2-zx^2)^2$ is unaltered in value by an interchange of the letters x, y, z.

Solution by A. P. Shepherd; H. Murphy; J. R. Wilson, M.A.; *and many others.*

$$(y^2+z^2)(z^2+x^2)(x^2+y^2) = \begin{vmatrix} y, & -z \\ z, & y \end{vmatrix} \times \begin{vmatrix} x, & -y \\ y, & x \end{vmatrix} \times \begin{vmatrix} z, & -x \\ x, & z \end{vmatrix}$$

$$= \begin{vmatrix} y, & -z \\ z, & y \end{vmatrix} \times \begin{vmatrix} x(y+z), & x^2-yz \\ yz-x^2, & x(y+z) \end{vmatrix}$$

$$= \begin{vmatrix} xyz+y^2x-zx^2+yz^2, & xz^2+xyz+yx^2-y^2z \\ y\cdot z-x^2y-xyz-xz^2, & yz^2-x^2z+y^2z+xyz \end{vmatrix}$$

$$= (xyz+xy^2+yz^2-zx^2)^2+(xyz+x^2y-y^2z+z^2x)^2.$$

By multiplying together the three determinants differently, we obtain the two expressions formed from the given one by interchanging x, y, z. Thus these expressions are all equal.

Otherwise:—Denote the given expression by a^2+b^2; then

$$ax+by = (x^2+y^2)(xy+z^2), \quad ay-bx = 2(x-y)(x^2+y^2).$$

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Therefore, squaring and adding, we have

$$a^2+b^2=[z^2(x-y)^2+(xy+z^2)^2](x^2+y^2)=(y^2+z^2)(z^2+x^2)(x^2+y^2),$$

which is a symmetrical function of x, y, z.

[By others the expression is thrown into the following forms:—

$$2x^2y^2z^2+y^2z^2(y^2+z^2)+z^2x^2(z^2+x^2)+x^2y^2(x^2+y^2);$$

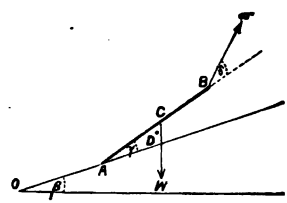
$$x^2y^2z^2\{(x^2+y^2+z^2)(x^{-2}+y^{-2}+z^{-2})-1\}.]$$

4256. (Proposed by W. Siverly.)—A uniform rod of length $2a$ and weight W, rests with its lower end on a smooth inclined plane of inclination β (the rod being inclined to the plane at an angle γ), and is supported at a distance b from the lower end by a vertical prop, and is kept in equilibrium by a string fastened to the upper end and passing over a pulley; a weight w being attached to the other end of the string which hangs freely, the string making an angle δ with the rod. Show that

$$\frac{W}{w}=\frac{b\sin\beta+a\cos(\beta+\gamma)\sin\gamma}{b\cos(\delta+\gamma)+2a\sin\delta\sin\gamma}.$$

I. *Solution by* S. Forde; C. Leudesdorf; *and others.*

Taking for axes the horizontal and vertical lines through O, the base of the inclined plane, and putting h for the x-coordinate of A, we have



for C, $x=h+a\cos(\beta+\gamma)$,
$y=h\tan\beta+a\sin(\beta+\gamma)$,

for B, $x=h+2a\cos(\beta+\gamma)$,
$y=h\tan\beta+2a\sin(\beta+\gamma)$,

for D, $x=h+b\cos(\beta+\gamma)$,
$y=h\tan\beta+b\sin(\beta+\gamma)$.

Now let the rod be slightly displaced, but so as to remain still in contact with the plane and the prop; then, differentiating the coordinates of D, we have

$$dx=0=dh+db\cos(\beta+\gamma)-bd\gamma\sin(\beta+\gamma),$$

$$dy=0=dh\tan\beta+db\sin(\beta+\gamma)+bd\gamma\cos(\beta+\gamma),$$

whence
$$bd\gamma=dh\frac{\sin\gamma}{\cos\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and
$$\frac{w}{W}=\frac{\text{displacement of C vertically}}{\text{displacement of B along B}w}=$$

$$\frac{dh\tan\beta+a\cos(\beta+\gamma)\,d\gamma}{[dh-2a\sin(\beta+\gamma)d\gamma]\cos(\beta+\gamma+\delta)+[dh\tan\beta+2a\cos(\beta+\gamma)d\gamma]\sin(\beta+\gamma+\delta)}$$

$$=\frac{b\sin\beta+a\cos(\beta+\gamma)\sin\gamma}{b\cos(\gamma+\delta)+2a\sin\gamma\sin\delta},\ \text{by (1).}$$

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II. *Solution by the* PROPOSER.

Let P and Q be the reactions on the rod of the prop and plane respectively; then resolving forces parallel and perpendicular to the rod, and taking moments about its lower end, we have

$$Q \sin\gamma = W \sin(\beta+\gamma) - w\cos\delta \quad \ldots\ldots (1);$$

$$Q \cos\gamma = W \cos(\beta+\gamma) + w\sin\delta - P \quad \ldots\ldots (2);$$

$$Wa\cos(\beta+\gamma) = 2aw\sin\delta + Pb \quad \ldots\ldots (3).$$

Dividing (1) by (2), $$\frac{\sin\gamma}{\cos\gamma} = \frac{W\sin(\beta+\gamma) - w\cos\delta}{W\cos(\beta+\gamma) - w\sin\delta - P} \quad \ldots\ldots (4).$$

Substituting value of P from (3) in (4), we have the result in the Question.

4461. (Proposed by Professor WOLSTENHOLME, M.A.)—A triangle ABC is circumscribed about a fixed ellipse, of focus S, such that the angles SBC, SCA, SAB are all equal: prove that each of them $= \sin^{-1}(\frac{1}{2}a^{-1}b)$, and that the angular points of the triangle lie on one of two fixed circles whose radius is $2a^2b^{-1}$; $2a$, $2b$ being the axes of the ellipse.

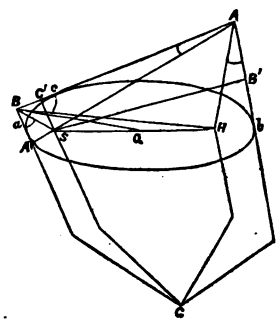
I. *Solution by the* PROPOSER.

If S, H be the foci of an ellipse, ABC a circumscribed triangle, such that

$$\angle SBC = \angle SCA = \angle SAB = \theta,$$

then will, of course,

$$\angle HBA = \angle HCB = \angle HAC = \theta.$$



Also $$\frac{SA}{SB} = \frac{\sin(B-\theta)}{\sin\theta}, \quad \frac{SB}{SC} = \frac{\sin(C-\theta)}{\sin\theta},$$

$$\frac{SC}{SA} = \frac{\sin(A-\theta)}{\sin\theta};$$

so that the equation for determining θ is

$$\sin(A-\theta)\sin(B-\theta)\sin(C-\theta) = \sin^3\theta,$$

or $$\sin(2A+\theta)+\sin(2B+\theta)+\sin(2C+\theta) - \sin 3\theta = 3\sin\theta - \sin 3\theta,$$

or $$\tan\theta = \frac{\sin 2A+\sin 2B+\sin 2C}{3-\cos 2A-\cos 2B-\cos 2C} = \frac{\sin A\sin B\sin C}{1+\cos A\cos B\cos C}.$$

From S draw SA′, SB′, SC′ perpendiculars to the sides, then A′, B′, C′ lie on the auxiliary circle of the ellipse, whose radius is a, But SA, SB, SC are inclined respectively to SC′, SA′, SB′ at the same angle $\frac{1}{2}\pi - \theta$; hence A, B, C will lie on a circle whose radius is

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a cosec θ, or $a = R \sin\theta$. The centre O of this circle will be on the minor axis of the ellipse, and the angle $OSH = \frac{1}{2}\pi - \theta$. Again, the rectangle under the perpendiculars from S, H on BC $= b^2 = SB \cdot HC \sin^2\theta$, and

$$\frac{SB}{AB} = \frac{\sin\theta}{\sin B}, \quad \frac{HC}{AC} = \frac{\sin\theta}{\sin C},$$

whence $b^2 = 4R^2 \sin^4\theta$, or $b = 2R\sin^2\theta$, or $\sin\theta = \frac{b}{2a}$, and $R = \frac{2a^2}{b}$;

so that the circle ABC is one of two fixed circles, whose centres are on the minor axis at a distance $(a^2 - b^2)^{\frac{1}{2}} \cot\theta$, or $(a^2 - b^2)^{\frac{1}{2}} (4a^2 - b^2)^{\frac{1}{2}} b^{-1}$ from the centre. It follows readily that

$$\frac{\sin(A-\theta)}{\sin^3 A} = \frac{\sin(B-\theta)}{\sin^3 B} = \frac{\sin(C-\theta)}{\sin^3 C} = \frac{\sin\theta}{\sin A \sin B \sin C}$$

$$= \frac{\cos\theta}{1 + \cos A \cos B \cos C} = \frac{\sin(A+\theta)}{\sin A(\sin^2 B + \sin^2 C)}$$

$$= \frac{\sin(B+\theta)}{\sin B(\sin^2 C + \sin^2 A)} = \frac{\sin(C+\theta)}{\sin C(\sin^2 A + \sin^2 B)};$$

also that $$\frac{SA}{\sin^2 B \sin C} = \frac{SB}{\sin^2 C \sin A} = \frac{SC}{\sin^2 A \sin B} = \frac{HA}{\sin B \sin^2 C}$$

$$= \frac{HB}{\sin C \sin^2 A} = \frac{HC}{\sin A \sin^2 B} = \frac{2R\sin\theta}{\sin A \sin B \sin C}.$$

Hence $\triangle SAB = \triangle HAC$, &c., or if (X, Y, Z), (X′, Y′, Z′) be areal coordinates of the foci referred to the triangle ABC, X = Z, Y = X′, Z = Y′, so that the foci are given in areal coordinates by $(b^{-2} : c^{-2} : a^{-2})$, $(c^{-2} : a^{-2} : b^{-2})$, and the centre by $a^2(b^2 + c^2) : b^2(c^2 + a^2) : c^2(a^2 + b^2)$. The reciprocals of the diameters parallel to the sides are therefore as

$$a(b^2 + c^2) : b(c^2 + a^2) : c(a^2 + b^2),$$

whence, if a, b, c be the points of contact, we have

$$\frac{Ab}{Ac} = \frac{c(a^2 + b^2)}{b(c^2 + a^2)}, \text{ say } Ab = \lambda c(a^2 + b^2), \; Ac = \lambda b(c^2 + a^2);$$

and similarly, $$Bc = \mu a(b^2 + c^2), \quad Ba = \mu c(a^2 + b^2);$$

$$Ca = \nu b(c^2 + a^2), \quad Cb = \nu a(b^2 + c^2);$$

whence $$a = \mu c(a^2 + b^2) + \nu b(c^2 + a^2), \quad b = \nu a(b^2 + c^2) + \lambda c(a^2 + b^2),$$

$$c = \lambda b(c^2 + a^2) + \mu a(b^2 + c^2);$$

therefore $$b^2(c^2 + a^2) + c^2(a^2 + b^2) - a^2(b^2 + c^2) = 2\lambda bc(a^2 + b^2)(a^2 + c^2) = 2b^2c^2,$$

or $$\lambda = \frac{bc}{(a^2 + b^2)(a^2 + c^2)}, \text{ &c.}$$

Hence $$\frac{Ba}{aC} = \frac{\mu c(a^2 + b^2)}{\nu b(c^2 + a^2)} = \frac{ac^2(a^2 + b^2)(a^2 + c^2)}{ab^2(b^2 + a^2)(a^2 + b^2)} = \frac{c^2}{b^2};$$

hence a lies on the straight line joining A to the intersection of the tangents at B, C to the circle ABC.

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II. *Solution by* R. TUCKER, M.A.

Let θ be the required angle; then

$$\sin\theta = \frac{\gamma}{SA} = \frac{\alpha}{SB} = \frac{\beta}{SC}$$

and $SA^2 = (\beta^2+\gamma^2+2\beta\gamma\cos A)\operatorname{cosec}^2 A = [\beta,\gamma]$ say ..... (1),

with similar expressions for SB and SC;

therefore $\sin^6\theta = \dfrac{\alpha^2\beta^2\gamma^2}{(\beta,\gamma)(\gamma,\alpha)(\alpha,\beta)} = \dfrac{\alpha^2\beta^2\gamma^2\sin^2 A\sin^2 B\sin^2 C}{4a^2(a\gamma\sin B+\ldots+\ldots)^2}$

$$= \frac{(a'\alpha)^2(b'\beta)^2(c'\gamma)^2}{4a^2\times 16R^4(b'a\gamma+\ldots+\ldots)^2}$$

where a', b', c' are the sides of the triangle;

that is, $\sin^6\theta = \dfrac{b^4}{16a^2R^2}$ (see *Messenger of Mathematics*, vol. I. p. 99)... (2).

Now $\angle ASC = 180°-A$, $\angle ASB = 180°-B$, $\angle BSC = 180°-C$,

and $\sin\theta = \dfrac{SA\sin A}{b'} = \dfrac{SB\sin B}{c'} = \dfrac{SC\sin C}{a'}$;

that is, $\sin^3\theta = \left(\dfrac{\sin A}{a'}\right)^3 SA.SB.SC$; therefore, by (1), $\alpha\beta\gamma = 8R^3\sin^6\theta$.

Similarly, by a property of the foci, we have

$$\frac{b^6}{\alpha\beta\gamma} = 8R^3\sin^6\theta;\ \text{therefore } b = 2R\sin^2\theta \ \ldots\ldots\ldots\ldots (3).$$

From (2) and (3) we get $\sin\theta = \dfrac{b}{2a}$, $R = \dfrac{2a^2}{b}$.

We may notice that $\alpha\beta\gamma = b^3$, also $\cot\theta = \cot A+\cot B+\cot C$;

this latter property being obtained by multiplying together the equations

$$\sin A\cot\theta-\cos A = \frac{\beta}{\gamma},\qquad \sin B\cot\theta-\cos B = \frac{\gamma}{\alpha},$$

$$\sin C\cot\theta-\cos C = \frac{\alpha}{\beta}.$$

The symmetry of the figure shows that there must be two circles.

4458. (Proposed by Professor CAYLEY.)—Find (1) the intersections of the two quartic curves

$$\lambda(ab-xy)^2 = abx(a-y)(b-y),\quad \mu(ab-xy)^2 = aby(a-x)(b-x);$$

and (2) trace the curves in some particular cases; for instance, when $a=1$, $b=2$, $\lambda=1$, $\mu=-2$.

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Solution by the PROPOSER.

1. The 16 intersections are made up as follows: 5 points at infinity on the line $x=0$, 5 at infinity on the line $y=0$, the two points $(x=a, y=b)$, $(x=b, y=a)$, and 4 other points, $16 = 5+5+2+4$. As to the points at infinity, observe that, as regards the first curve, the point at infinity on the line $x=0$ is a flecnode having this line for a tangent to the flecnodal branch; and, as regards the second curve, the same point is a cusp, having this line for its tangent; hence the point in question counts as $2+3$, $= 5$ intersections; and the like as to the point at infinity on the line $y=0$. It remains to find the coordinates of the 4 points of intersection. Assume $xy = ab\omega$, then the equations become

$$\lambda(1-\omega)^2 = x + \omega y - (a+b)\omega, \quad \mu(1-\omega)^2 = \omega x + y - (a+b)\omega;$$

and hence, eliminating successively y and x, the factor $1-\omega$ divides out [this factor belongs to the points $(x=a, y=b)$, $(x=b, y=a)$ for which obviously $\omega=1$], and the equations become

$$(\lambda-\mu\omega)(1-\omega) + (a+b)\omega = (1+\omega)x, \quad (\mu-\lambda\omega)(1-\omega) + (a+b)\omega = (1+\omega)y.$$

Multiplying these two equations together, and substituting for xy its value $ab\omega$, we find

$$\left\{(\lambda-\mu\omega)(\mu-\lambda\omega) + (a+b)(\lambda+\mu)\omega\right\}(1-\omega)^2 + (a+b)^2\omega^2 - (1+\omega)^2\omega ab = 0.$$

Write, for shortness, $p = (\lambda+\mu)(a+b) - \lambda^2 - \mu^2$, then, dividing by ω^2, and writing $\omega + \frac{1}{\omega} = \Omega$, the equation is

$$(\lambda\mu\Omega + p)(\Omega - 2) + (a+b)^2 - ab(\Omega+2) = 0;$$

viz., this is a quadric equation for Ω. But, instead of Ω, it is convenient to introduce the quantity θ, $= \frac{\Omega-2}{\Omega+2}$, $= \left(\frac{\omega-1}{\omega+1}\right)^2$. The equation thus becomes

$$\left\{2\lambda\mu\frac{1+\theta}{1-\theta} + p\right\}\frac{4\theta}{1-\theta} + (a+b)^2 - ab\frac{4}{1-\theta} = 0,$$

or

$$\left\{2\lambda\mu(1+\theta) + p(1-\theta)\right\}4\theta + (a+b)^2(1-\theta)^2 - 4ab(1-\theta) = 0,$$

or

$$\theta^2\left\{(a+b)^2 - 4(p-2\lambda\mu)\right\} + \theta\left\{-2a^2 - 2b^2 + 4(p+2\lambda\mu)\right\} + (a-b)^2 = 0;$$

viz., substituting for p its values, this is

$$\theta^2(a+b-2\lambda-2\mu)^2 + 2\theta\left\{-a^2-b^2+2(\lambda+\mu)(a+b) - 2(\lambda-\mu)^2\right\}^2 + (a-b)^2 = 0;$$

or if we write $A = a^2 - 2a(\lambda+\mu) + (\lambda-\mu)^2$, $B = b^2 - 2b(\lambda+\mu) + (\lambda-\mu)^2$, this is

$$\theta^2(a+b-2\lambda-2\mu)^2 - 2(A+B)\theta + (a-b)^2 = 0,$$

whence

$$\left\{(a-b)^2 - (A+B)\theta\right\}^2 = \theta^2\left\{(A+B)^2 - (a-b)^2(a+b-2\lambda-2\mu)^2\right\}$$
$$= \theta^2\left\{(A+B)^2 - (A-B)^2\right\} = 4AB\theta^2;$$

viz., taking for convenience the sign $-$ on the right hand side, this is

$$(a-b)^2 - (A+B)\theta = -2\theta\sqrt{AB}; \quad \text{and we have thus}$$

$$\theta = \frac{(a-b)^2}{(\sqrt{A}-\sqrt{B})^2}, \text{ that is } \theta, = \frac{\omega-1}{\omega+1} = \frac{a-b}{\sqrt{A}-\sqrt{B}}; \quad \omega = \frac{\sqrt{A}-\sqrt{B}+a-b}{\sqrt{A}-\sqrt{B}-a+b}.$$

We may write

$$x = \mu(\omega-1) + \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1},$$

$$y = \lambda(\omega-1) + \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1};$$

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[whence also $x-y=(\mu-\lambda)(\omega-1)$, as is also seen at once from the original equations]; then we have

$$\tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1}=\frac{\tfrac{1}{2}(a-b)(a+b-2\lambda-2\mu)}{\sqrt{A}-\sqrt{B}}$$
$$=\frac{\tfrac{1}{2}(A-B)}{\sqrt{A}-\sqrt{B}},\ =\tfrac{1}{2}(\sqrt{A}+\sqrt{B});$$

and the values are

$$x=\frac{2\mu(a-b)}{\sqrt{A}-\sqrt{B}-a+b}+\tfrac{1}{2}(\sqrt{A}+\sqrt{B}+a+b),$$
$$=\frac{(a-b)(\mu-\lambda)+b\sqrt{A}-a\sqrt{B}}{\sqrt{A}-\sqrt{B}-a+b},$$
$$y=\frac{2\lambda(a-b)}{\sqrt{A}-\sqrt{B}-a+b}+\tfrac{1}{2}(\sqrt{A}+\sqrt{B}+a+b),$$
$$=\frac{(a-b)(\lambda-\mu)+b\sqrt{A}-a\sqrt{B}}{\sqrt{A}-\sqrt{B}-a+b},$$

which may be expressed in the more simple form

$$x=\frac{1}{4\lambda}(a+\lambda-\mu+\sqrt{A})(b+\lambda-\mu+\sqrt{B}),$$
$$y=\frac{1}{4\mu}(a-\lambda+\mu+\sqrt{A})(b-\lambda+\mu+\sqrt{B}),$$

the transformations depending on the identity

$$\frac{8\lambda\mu(a-b)}{\sqrt{A}-\sqrt{B}-a+b}=ab-(\lambda+\mu)(a+b)+(\lambda-\mu)^2+\sqrt{A}(b-\lambda-\mu)$$
$$+\sqrt{B}(a-\lambda-\mu)+\sqrt{AB},$$

which is easily verified. Of course, since the signs of $\sqrt{A}$, $\sqrt{B}$ are arbitrary, we have 4 systems of values of (x, y), which is right.

In the original equations, for a, b, λ, μ, x, y, write $1, k^{-2}, \lambda^2, -\mu^2, x^2, -y^2$; then the equations become

$$\lambda^2(1+k^2x^2y^2)^2=x^2(1+y^2)(1+k^2y^2),\quad \mu^2(1+k^2x^2y^2)^2=y^2(1-x^2)(1-k^2x^2),$$

and we thence have

$$\lambda+\mu i=\frac{x\sqrt{(1+y^2)(1+k^2y^2)}+iy\sqrt{(1+x^2)(1-k^2x^2)}}{1+k^2x^2y^2};$$

viz., assuming $x=\operatorname{sn}\alpha$ ($\operatorname{sinam}\alpha$), $iy=\operatorname{sn}i\beta$, this is $\lambda+\mu i=\operatorname{sn}(\alpha+\beta i)$; viz., the problem is (for a given modulus k, assumed as usual to be real, positive, and less than 1) to reduce a given imaginary quantity $\lambda+\mu i$ to the form $\operatorname{sn}(\alpha+\beta i)$. The proper solution is that in which the signs of the radicals are each $-$, viz., it may in this case be shown that the value of x^2 is positive and less than 1, that of y^2 positive. The values thus are

$$x^2=\frac{1}{4\lambda^2}(1+\lambda^2+\mu^2-\sqrt{A})\left(\frac{1}{k^2}+\lambda^2+\mu-\sqrt{B}\right)$$
$$y^2=\frac{1}{4\mu^2}(1-\lambda^2-\mu^2-\sqrt{A})\left(\frac{1}{k^2}-\lambda^2-\mu^2-\sqrt{B}\right),$$

where $A=1-2\lambda^2+2\mu^2+(\lambda^2+\mu^2)^2$, $B=\frac{1}{k^2}-\frac{2}{k^2}\lambda^2+\frac{2}{k^2}\mu^2+(\lambda^2+\mu^2)^2$.

The solution is really equivalent to that given by Richelot (Crelle, t. 45, 1853, p. 225); to partially verify this, observe that writing σ, τ for

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Richelot's $\tan\frac{1}{2}\phi$, $\tan\frac{1}{2}\psi$, we have

$$\left(\sigma+\frac{1}{\sigma}\right)\lambda = 1+\lambda^2+\mu^2, \text{ giving } \left(\sigma-\frac{1}{\sigma}\right)\lambda = -\sqrt{A},$$

$$\left(\tau+\frac{1}{\tau}\right)\frac{\lambda}{k} = \frac{1}{k^2}+\lambda^2+\mu^2, \text{ giving } \left(\tau-\frac{1}{\tau}\right)\frac{\lambda}{k} = -\sqrt{B};$$

whence $2\sigma\lambda = 1+\lambda^2+\mu^2-\sqrt{A}$, $2\tau\frac{\lambda}{k} = \frac{1}{k^2}+\lambda^2+\mu^2-\sqrt{B}$,

or the above value of x^2 is $= k^{-1}\sigma\tau$, agreeing with his; the value of y^2 is however, presented under a somewhat different form.

2. The curves are

$$(2-xy)^2 = 2x(1-y)(2-y), \quad -(2-xy)^2 = y(1-x)(2-x) \ldots (1, 2),$$

each passing through the points (1, 2) and (2, 1); the four points of intersection found by the foregoing general theory are all real, viz., these are

$x=\frac{1}{8}(2+\sqrt{3})(5+\sqrt{17})$,	$y=-\frac{1}{8}(-1+\sqrt{3})(-1+\sqrt{17})$,	say	$+17{\cdot}00$ &	$-0{\cdot}57$
$-\sqrt{3}$,	$+\sqrt{17}$	,,	,,	$+1{\cdot}65$	$+0{\cdot}94$
$+\sqrt{3}$,	$-\sqrt{17}$	,,	,,	$+1{\cdot}22$	$+2{\cdot}13$
$-\sqrt{3}$,	$-\sqrt{17}$	,,	,,	$-0{\cdot}12$	$-3{\cdot}49$

The equation of the curve (1) may also be written in the forms

$$y^2(x^2-2x)+2yx-4x+4 = 0, \quad x^2y^2+x(-2y^2+2y-4)+4 = 0.$$

The original form shows that if y is between 1 and 2, x is negative—(but by a further examination it appears that there is not in fact any branch of the curve between these limits of y)—but y being outside these limits, then x is positive; in fact, the whole curve lies on the positive side of the axis of y. And then the inspection of the first quadric equation shows that the lines $x=0$ and $x=2$ are each an asymptote.

The point at infinity on the axis of y is in fact a flecnode, the tangent to the flecnodal branch being $x=0$, and that of the ordinary branch $x=2$.

Similarly, from the second quadric equation, it appears that the line $y=0$ is an asymptote; the point at infinity on the axis of x is in fact a cusp, the axis in question $y=0$ being the cuspidal tangent.

The equation of the curve (2) may also be written in the forms

$$x^2y^2+(x^2-7x+2)y+4 = 0, \quad (y^2+y)x^2-7yx+2y+4 = 0.$$

The original form shows that if x is between 1 and 2, y is positive; but that x being beyond these limits, y is negative; and as regards the first case, x between 1 and 2, we at once establish the existence of an oval, meeting the line $y=1$ in the points $x=2$ and $\frac{3}{2}$, and the line $y=2$ in the points $x=1$ and $\frac{4}{5}$; it is further easy to see that the horizontal tangents of the oval are $y = \frac{1}{16}(25\pm\sqrt{113})$, = say $2{\cdot}2$ and $0{\cdot}9$.

The remainder of the curve lies wholly below the line $y=0$. The first quadric equation shows the asymptote $x=0$; the point at infinity on the axis of y is in fact a cusp having the axis itself for the cuspidal tangent. The second quadric equation shows the asymptotes $y=0$, $y=-1$; the point at infinity on the axis of x is in fact a flecnode having the line $y=0$ for the tangent to the flecnodal branch, and $y=-1$ for that of the other branch. It is further seen that there are two vertical tangents $x = \frac{1}{8}(11\pm\sqrt{113}) = 10{\cdot}8$ or $0{\cdot}2$; the former of these touches a branch lying wholly between the two asymptotes $y=0$, $y=-1$; the latter one of the branches belonging to the cuspidal asymptote $x=0$; this last branch cuts the asymptote $x=0$ at $y=-2$, and then cutting the asymptote

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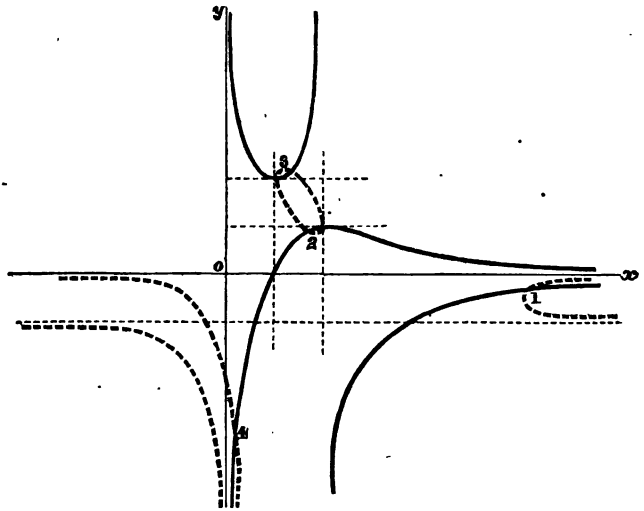
$y = -1$ and $x = -\frac{2}{7}(= -0{\cdot}3)$, goes on to touch at infinity the asymptote $y = 0$. It is now easy to trace the curve.



The figure shows the two curves. The curve (1) is shown by a continuous line, the curve (2) by a dotted line; the points 1, 2, 3, 4 show the above mentioned four intersections of the curves; the point 1 and the dotted branch through it are of necessity drawn considerably out of their true positions; viz., as above appearing, the x coordinate of 1 is $= 17{\cdot}00$, and the equation of the vertical tangent to the branch is $x = 10{\cdot}8$.

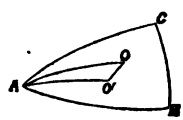
4368. (Proposed by the Rev. Dr. Booth, F.R.S.)—A spherical triangle is inscribed in one circle, and circumscribed to another circle of a sphere whose spherical radii are r and R, and Δ is the arc of a great circle joining their centres; show (1) that

$$\sin^2\Delta = \cos^2 \mathrm{R} \cos^2 r \,(\tan^2 \mathrm{R} - 2 \tan \mathrm{R} \tan r)\,;$$

and (2) that when the sphere becomes a plane, then $\Delta^2 = \mathrm{R}^2 - 2\mathrm{R}r$.

Solution by Asher B. Evans, M.A.

Let O, O′ be the poles of the circumscribed and inscribed circles. Put



$$\mathrm{S} = \tfrac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C}), \quad s = \tfrac{1}{2}(a+b+c),$$
$$n^2 = \sin s \sin(s-a) \sin(s-b) \sin(s-c)\,;$$

then

$$\cos\Delta = \cos \mathrm{AO} \cos \mathrm{AO}' + \sin \mathrm{AO} \sin \mathrm{AO}' \cos \mathrm{OAO}' \ldots\ldots\ldots (1).$$

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Since $OAB = S-C$, $OAO' = \frac{1}{2}(B-C)$;

moreover $\cos AO' = \cos r \cos(s-a)$ and $\sin AO' = \dfrac{\sin r}{\sin \frac{1}{2}A}$;

whence from (1) we find

$$\frac{\cos\Delta}{\cos R \sin r} = \cot r \cos(s-a) + \tan R \cdot \frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A} \quad \ldots\ldots\ldots (2).$$

But $\cot r = \dfrac{\sin s}{n}$, $\tan R = \dfrac{2}{n} \sin\frac{1}{2}a \sin\frac{1}{2}b \sin\frac{1}{2}c$,

and by Gauss's Equations $\dfrac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A} = \dfrac{\sin\frac{1}{2}(b+c)}{\sin\frac{1}{2}a}$;

therefore (2) becomes

$$\frac{\cos\Delta}{\cos R \sin r} = \frac{\sin s \cos(s-a) + 2\sin\frac{1}{2}b \sin\frac{1}{2}c \sin\frac{1}{2}(b+c)}{n}$$

$$= \frac{\sin a + \sin b + \sin c}{2n}.$$

Observing that $4n^2 = 1-\cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c$,

and that $4[\sin s + 2\sin\frac{1}{2}a \sin\frac{1}{2}b \sin\frac{1}{2}c]^2$

$= 2(1 + \sin a \sin b + \sin a \sin c + \sin b \sin c - \cos a \cos b \cos c)$,

we readily find, from the preceding equation,

$$\left(\frac{\cos\Delta}{\cos R \sin r}\right)^2 - 1 = \left(\frac{\sin s + 2\sin\frac{1}{2}a \sin\frac{1}{2}b \sin\frac{1}{2}c}{n}\right)^2 = (\cot r + \tan R)^2;$$

therefore $\cos^2\Delta = \cos^2 R \sin^2 r [1 + (\cot r + \tan R)^2]$,

and $\sin^2\Delta = \cos^2 R \cos^2 r (\tan^2 R - 2\tan R \tan r)$ ..... (3).

The angles which Δ, R, r subtend at the centre of the sphere, expressed in arcs are $\frac{\Delta}{\rho}$, $\frac{R}{\rho}$, $\frac{r}{\rho}$, ρ being the radius of the sphere. When ρ becomes infinite, the spherical triangle becomes a plane triangle, and $\frac{\Delta}{\rho}$, $\frac{R}{\rho}$, $\frac{r}{\rho}$ being infinitely small,

$$\sin\frac{\Delta}{\rho} = \frac{\Delta}{\rho}, \quad \tan\frac{R}{\rho} = \frac{R}{\rho}, \quad \tan\frac{r}{\rho} = \frac{r}{\rho}, \quad \cos\frac{R}{\rho} = 1, \quad \cos\frac{r}{\rho} = 1,$$

and (3) gives $\Delta^2 = R^2 - 2Rr$.

If r_1, r_2, r_3 represent the radii of the circles inscribed in A'BC, B'CA, C'AB; and Δ_1, Δ_2, Δ_3 the distance of their centres from Q, we shall find by a similar process,

$$\sin^2\Delta_1 = \cos^2 R \cos^2 r_1 (\tan^2 R + 2\tan R \tan r_1),$$
$$\sin^2\Delta_2 = \cos^2 R \cos^2 r_2 (\tan^2 R - 2\tan R \tan r_2),$$
$$\sin^2\Delta_3 = \cos^2 R \cos^2 r_3 (\tan^2 R + 2\tan R \tan r_3);$$

and in the plane $\Delta_1^2 = R^2 + 2Rr_1$, &c.

4098. (Proposed by the EDITOR.)—A straight line of constant length c moves with its ends on the curve of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$

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show (1) that the locus of a point which divides the chord into the two parts λc, μc, which are in the constant ratio $\lambda : \mu$, is

$$a^2y\,(\lambda\mu z^2-\lambda^2\mu^2b^2c^2)^{\frac{1}{2}}+b^2x\,(\lambda^2\mu^2a^2c^2-\lambda\mu z^2)^{\frac{1}{2}}=\tfrac{1}{2}\,(\lambda-\mu)\,aez^2,$$

where $z^2=a^2b^2-a^2y^2-b^2x^2$; (2) that when the chords are bisected, the locus becomes $\frac{c^2}{4}\left(\frac{a^2y^2+b^2x^2}{a^4y^2+b^4x^2}\right)+\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; (3) that when the ellipse becomes a circle, the locus is $x^2+y^2=a^2-\lambda\mu c^2$; and (4) that the area of the locus is always $\pi\,(ab-\lambda\mu c^2)$.

Solution by A. M. Nash, B.A.; Christine Ladd; A. B. Evans, M.A.; *and others.*

1. Let $(\xi-x)\sec\theta=(\eta-y)\operatorname{cosec}\theta=r$ be the equation of the line in any position, (x, y) being the corresponding point of the locus; then, substituting for ξ and η in the equation of the ellipse, we get, for the determination of the lengths of the parts into which (x, y) divides the chord,

$$(a^2\sin^2\theta+b^2\cos^2\theta)\,r^2+2\,(b^2x\cos\theta+a^2y\sin\theta)\,r-z^2=0 \quad\ldots\ldots\ldots (1).$$

Put, by hypothesis, these lengths are $\pm\lambda c$, $\mp\mu c$; therefore (1) must be identical with

$$r^2\pm c\,(\lambda-\mu)\,r-\lambda\mu c^2=0 \quad\ldots\ldots\ldots (2);$$

therefore
$$a^2\sin^2\theta+b^2\cos^2\theta=\frac{2\,(b^2x\cos\theta+a^2y\sin\theta)}{\pm c\,(\lambda-\mu)}=\frac{z^2}{\lambda\mu c^2};$$

therefore
$$\sin^2\theta=\frac{z^2-\lambda\mu b^2c^2}{\lambda\mu c^2a^2e^2}, \text{ and } \cos^2\theta=\frac{\lambda\mu a^2c^2-z^2}{\lambda\mu c^2a^2e^2}.$$

But
$$b^2x\cos\theta+a^2y\sin\theta=\frac{\pm\,(\lambda-\mu)\,z^2}{2\lambda\mu c},$$

therefore
$$b^2x\,(\lambda\mu a^2c^2-z^2)^{\frac{1}{2}}+a^2y\,(z^2-\lambda\mu b^2c^2)^{\frac{1}{2}}=\frac{(\lambda-\mu)\,aez^2}{2\,(\lambda\mu)^{\frac{1}{2}}} \quad\ldots\ldots (3).$$

The ambiguity of sign being implied by the radical sign, the equation of the required locus is therefore that given in the Question.

2. When the chords are bisected, $\lambda=\mu=\frac{1}{2}$, and (3) becomes

$$b^2x\,(a^2c^2-4z^2)^{\frac{1}{2}}+a^2y\,(4z^2-b^2c^2)^{\frac{1}{2}}=0, \text{ or } a^2b^2c^2\,(b^2x^2+a^2y^2)=4z^2\,(a^4y^2+b^4x^2),$$

therefore
$$\frac{c^2}{4}\left(\frac{b^2x^2+a^2y^2}{b^4x^2+a^4y^2}\right)=\frac{z^2}{a^2b^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}.$$

3. When the ellipse is a circle, $a=b$, $e=0$, therefore (3) becomes

$$x\,(\lambda\mu a^2c^2-z^2)^{\frac{1}{2}}+y\,(z^2-\lambda\mu a^2c^2)^{\frac{1}{2}}=0,$$

and therefore $z^2-\lambda\mu a^2c^2=0$; therefore the equation of the locus is that given in the Question.

4. By Holditch's Theorem (Bertrand's *Calcul Intégral*, p. 365), we have

$$\text{Area required} = \text{Area of ellipse} - \pi\,.\,\lambda c\,.\,\mu c = \pi\,(ab-\lambda\mu c^2).$$

[Holditch's Theorem was first given as Question 1928 in the *Diary* for 1857, under the *nom de plume* "Petrarch." Four solutions of it were published in the *Diary* for 1858, together with a generalization of the theorem by Mr. Woolhouse, the Editor, who moreover gave a further extension of it in the *Diary* for 1858.]

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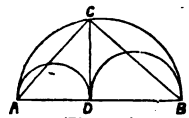
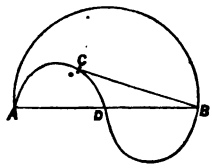
4423. (Proposed by T. T. WILKINSON, F.R.A.S.)—The diameter of a semicircle is divided into two unequal parts, on which two other semicircles are described; find a circle whose area shall be equal to the curvilinear area contained between the three semicircles.

Solution by H. MURPHY; BELLE EASTON; H. S. MONCK; *and others.*



(FIG. 1.)



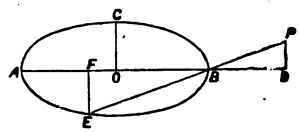
(FIG. 2.)

1. Let the three semicircles be drawn on the same side of the line (Fig. 1); then, since semicircles are similar figures, their areas are as the squares on their diameters, that is, as the squares on AB, AD, BD; and the curvilinear space between them is to any of the semicircles as $AB^2 - AD^2 - BD^2$ to the square on the corresponding line, *i.e.*, as $2AD \cdot BD$ to the corresponding square. Draw CD perpendicular to AB; then $2AD \cdot BD = 2CD^2$. Therefore the curvilinear space is to any of the semicircles as twice the square on CD to the corresponding square. But the latter ratio is that of twice the semicircle (*i.e.*, the whole circle) on CD to the corresponding semicircle. Therefore the curvilinear space is equal the circle on CD as diameter.

2. Let one semicircle be drawn on the opposite side of the line. Draw the tangent BC. It can be shown in a very similar way that the circle on BC as diameter is equal to the curvilinear space.

3693. (Proposed by WALTER SIVERLY.)—One end of a rod, whose length is equal to the major axis of an ellipse, is inserted through an opening at the extremity of the major axis, and made to pass around the curve; find the curve traced out by the other end of the rod.

Solution by ARTEMAS MARTIN.



Put $EP = c$, $BD = x$, $PD = y$, $OF = x_1$, and $EF = y_1$; then $FB = a + x_1$, $BP = (x^2 + y^2)^{\frac{1}{2}}$, and $BE = c - (x^2 + y^2)^{\frac{1}{2}}$. The triangles BDP, BFE are similar; therefore

$$x : (x^2 + y^2)^{\frac{1}{2}} = a + x_1 : c - (x^2 + y^2)^{\frac{1}{2}},$$

and $$y : (x^2 + y^2)^{\frac{1}{2}} = y_1 : c - (x^2 + y^2)^{\frac{1}{2}},$$

whence $$x_1 = \frac{cx}{(x^2 + y^2)^{\frac{1}{2}}} - (x + a), \quad \text{and} \quad y_1 = \frac{cy}{(x^2 + y^2)^{\frac{1}{2}}} - y.$$

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But $$a^2y_1^2 + b^2x_1^2 = a^2b^2;$$

therefore, by substitution, we have, for the equation of the required curve,

$$c^2(a^2y^2 + b^2x^2) - 2c\left\{a^2y^2 + b^2x(x+a)\right\}(x^2+y^2)^{\frac{1}{2}}$$
$$= \left\{a^2b^2 - a^2y^2 - b^2(x+a)^2\right\}(x^2+y^2) \quad \ldots\ldots\ldots\ldots (1).$$

When $c = a$, the equation becomes

$$a^2(a^2y^2 + b^2x^2) - 2a\left\{a^2y^2 + b^2x(x+a)\right\}(x^2+y^2)^{\frac{1}{2}}$$
$$= \left\{a^2b^2 - a^2y^2 - b^2(x+a)^2\right\}(x^2+y^2) \quad \ldots\ldots\ldots\ldots (2).$$

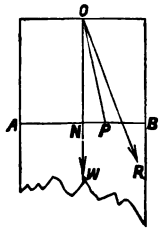
When $b = a$, this equation reduces to

$$4a^2\left\{y^2 + x(x+a)\right\}^2 = (x^2+y^2)\left\{y^2 + (x+a)^2\right\}^2 \quad \ldots\ldots\ldots\ldots (3).$$

4485. (Proposed by Professor TOWNSEND, F.R.S.)—The top of a vertical rectangular wall, of uniform material and construction, sustains the oblique thrust of an ordinary inclined roof, supposed uniformly distributed over its entire area; show (1) that the "line of pressure" of the entire force resulting from the propagated thrust and superincumbent weight, throughout any transverse section of the wall, is a rectangular hyperbola, passing through the middle point of the upper side of the section, and having its asymptotes horizontal and vertical; and (2) determine, for a given thrust of roof and weight of material, the minimum thickness which would enable the wall to be raised to any height without being overturned by the roof.

I. *Solution by* G. S. CARR.

1. Let the figure represent a cross section of the wall; R the roof's resultant thrust, passing through O, the middle point of the top of the wall.



Consider any horizontal section AB. It is subjected to the resultant pressures R and W, the latter being the weight of the superincumbent wall. Let the resultant of R and W intersect AB in P. Then P is a point in the line of pressure.

Take $PN = x$, $ON = y$ for the coordinates of P; X, Y for the horizontal and vertical components of R; then, since $W \propto y$, putting $W = cy$, the equation to the locus of P is

$$\frac{y}{x} = \frac{Y + cy}{X}, \quad \text{or} \quad cxy + Yx - Xy = 0,$$

which is a rectangular hyperbola situated as described.

2. Writing the equation in the form $Yx = y(X - cx)$, it appears that $x = c^{-1}X$ is the equation to the vertical asymptote. Consequently $2c^{-1}X$ is the thickness of wall requisite to ensure the line of pressure always being within it.

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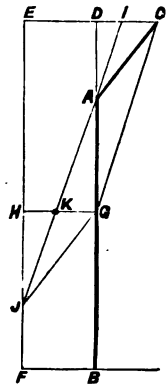
II. *Solution by the* PROPOSER.



Let AB be the medial altitude of the section, AC the line representing in magnitude and direction the thrust per unit of length of the roof on the same scale that AB does the weight per unit of length of the wall, CD the perpendicular from C on AB produced, DE the production of CD into the equal of its length, EF the altitude of E above the foot of the wall, GH any horizontal line intersecting AB and EF at G and H respectively, and I, J, K the points at which the right line through A parallel to CG intersects CD, EF, GH respectively; then, the force acting on the material of the wall along the line GH, arising from the propagated thrust of the roof and the superincumbent weight of the wall, being represented on a common scale, in magnitude and position, by the lines CA and AG; their resultant, equal and parallel to CG, is represented on the same scale, in magnitude and position, by the line AJ, the locus of whose intersection K with the line GH, supposed variable, is consequently the "line of pressure" required.

But since, by similar and equal triangles ADI and JHK, AI = JK, that locus, as above stated, is the rectangular hyperbola, passing through A, of which CD and EF are the asymptotes, and which consequently will never intersect the outer face of the wall, whatever be its height, provided it have a thickness at least = CE.

3891. (Proposed by Dr. HART.)—Find n numbers whose sum is a square, and the sum of their squares a biquadratic.

Solution by ASHER B. EVANS, M.A.

Let x, a_1y, a_2y ..... $a_{n-1}y$ represent the n numbers, and put

$$a_2+a_3+ \ldots\ldots +a_{n-1}=m \text{ and } a_2^2+a_3^2+ \ldots\ldots +a_{n-1}^2=n_1 \ldots\ldots (1).$$

Then a^4 being the biquadrate, we must have

$$x+(a_1+m)\,y=\square, \text{ and } x^2+(a_1^2+n_1)\,y^2=a^4 \ldots\ldots (2, 3).$$

Let $x=a^2-py$; then (3) will be satisfied, if

$$y=\frac{2a^2p}{a_1^2+n_1+p^2}; \text{ whence } x=a^2-py=\frac{a^2(a_1^2+n_1-p^2)}{(a_1^2+n_1+p^2)}.$$

These values of x and y, substituted in (2), give, after multiplying by $a^2(a^2+n_1+p^2)^2$, $(a_1^2+n_1)^2-p^4+2p(a_1+m)(a_1^2+n_1+p^2)$, which must be a square; let $(a_1^2+pa_1+b)$ be the root of this square; then, putting $2b=2pm+2n_1-p^2$, we find $a_1=\frac{\frac{5}{4}p^2+m^2-n_1-3pm}{3p-2m}$ ..... (4).

By giving suitable values to a_2, a_3 ..... a_{n-1}, we may find m and n from (1); then, by giving p and a suitable values, we shall obtain n positive integral numbers that will answer the conditions.

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4331. (Proposed by A. MARTIN.)—Find rational triangles whose sides, the radius of its inscribed circle, and the radii of "Malfatti's circles," shall all be rational numbers.

I. *Solution by the* PROPOSER.

Let a, b, c be the sides of the triangle, r the radius of the inscribed circle, and x, y, z the radii of "Malfatti's circles"; then it is known that the values of x, y, z are

$$\frac{r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}B)}{2(1+\tan\frac{1}{4}C)}, \quad \frac{r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}C)}{2(1+\tan\frac{1}{4}B)},$$

$$\frac{r(1+\tan\frac{1}{4}B)(1+\tan\frac{1}{4}C)}{2(1+\tan\frac{1}{4}A)};$$

and the values of a, b, c are

$$r(\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C), \quad r(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}C), \quad r(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B).$$

Take $\cot\frac{1}{4}C=3$, $\cot\frac{1}{4}B=4$; then

$$\cot\tfrac{1}{4}A=\tfrac{8}{9}, \quad \cot\tfrac{1}{2}A=\tfrac{17}{36}, \quad \cot\tfrac{1}{2}B=\tfrac{15}{8}, \quad \cot\tfrac{1}{2}C=\tfrac{4}{3},$$

$$x=\frac{55r}{96}, \quad y=\frac{88r}{135}, \quad z=\frac{15r}{22}; \quad a=\frac{231r}{72}, \quad b=\frac{250r}{72}, \quad c=\frac{289r}{72}.$$

Take $r=72$, then $a=231$, $b=250$, $c=289$; $x=41\frac{1}{4}$, $y=46\frac{14}{15}$, $z=49\frac{1}{11}$.

II. *Solution by* ASHER B. EVANS, M.A.

Let ABC be the triangle, r the radius of the inscribed circle, and ρ_1, ρ_2, ρ_3 the radii of Malfatti's circles. Then we have

$$a=\frac{r\cos\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C}, \quad b=\frac{r\cos\frac{1}{2}B}{\sin\frac{1}{2}A\sin\frac{1}{2}C}, \quad c=\frac{r\cos\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B};$$

$$\rho_1=\frac{r(1+\tan\frac{1}{4}B)(1+\tan\frac{1}{4}C)}{2(1+\tan\frac{1}{4}A)},$$

$$\rho_2=\frac{r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}C)}{2(1+\tan\frac{1}{4}B)}, \quad \rho_3=\frac{r(1+\tan\frac{1}{4}A)(1+\tan\frac{1}{4}B)}{2(1+\tan\frac{1}{4}C)} \ldots(1).$$

Let us first make the functions of A, B, C, in the foregoing values of $a, b, c, \rho_1, \rho_2, \rho_3$, rational. To this end let

$$\cot\tfrac{1}{2}A=\frac{m}{n}, \quad \cot\tfrac{1}{2}B=\frac{p}{q}, \quad \text{where } m^2+n^2=\square, \quad p^2+q^2=\square.$$

Employing the usual trigonometrical formulæ, we find

$$\cot\tfrac{1}{2}C=\frac{np+mq}{mp-nq}, \quad \cos\tfrac{1}{2}A=\frac{m}{(m^2+n^2)^{\frac{1}{2}}}, \quad \sin\tfrac{1}{2}A=\frac{n}{(m^2+n^2)^{\frac{1}{2}}},$$

$$\cos\tfrac{1}{2}B=\frac{p}{(p^2+q^2)^{\frac{1}{2}}}, \quad \sin\tfrac{1}{2}B=\frac{q}{(p^2+q^2)^{\frac{1}{2}}}, \quad \cot\tfrac{1}{2}C=\frac{np+mq}{\{(m^2+n^2)(p^2+q^2)\}^{\frac{1}{2}}},$$

$$\sin\tfrac{1}{2}C=\frac{mp-nq}{\{(m^2+n^2)(p^2+q^2)\}^{\frac{1}{2}}}, \quad \tan\tfrac{1}{2}A=\frac{n}{m+(m^2+n^2)^{\frac{1}{2}}},$$

$$\tan\tfrac{1}{2}B=\frac{q}{p+(p^2+q^2)^{\frac{1}{2}}}, \quad \tan\tfrac{1}{2}C=\frac{mp-nq}{np+mq+\{(m^2+n^2)(p^2+q^2)\}^{\frac{1}{2}}} \ldots\ldots(2).$$

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Equations (1) and (2) will give rational values to $a, b, c, \rho_1, \rho_2, \rho_3$, and thus furnish a general solution to the question, when m, n, p, q are so chosen as to satisfy the conditions $m^2+n^2 = \square$, $p^2+q^2 = \square$.

For a particular example let $m=4$, $n=3$, $p=12$, $q=5$; then $\cot \frac{1}{2}A = \frac{4}{3}$, $\cot \frac{1}{2}B = \frac{15}{8}$, $\cot \frac{1}{2}C = \frac{56}{33}$, $\cos \frac{1}{2}A = \frac{4}{5}$, $\cos \frac{1}{2}B = \frac{15}{17}$, $\cos \frac{1}{2}C = \frac{56}{65}$, $\sin \frac{1}{2}A = \frac{3}{5}$, $\sin \frac{1}{2}B = \frac{8}{17}$, $\sin \frac{1}{2}C = \frac{33}{65}$, $\tan \frac{1}{2}A = \frac{1}{2}$, $\tan \frac{1}{2}B = \frac{1}{3}$, $\tan \frac{1}{2}C = \frac{3}{11}$; and $a = \frac{476}{195}r$, $b = \frac{100}{33}r$, $c = \frac{56}{15}r$, $\rho_1 = \frac{63}{110}r$, $\rho_2 = \frac{20}{33}r$, $\rho_3 = \frac{24}{55}r$, where r may be any positive rational number. If we are restricted to integers, we may take $r = 6930$, then $a = 28392$, $b = 21000$, $c = 25872$, $\rho_1 = 3969$, $\rho_2 = 4900$, $\rho_3 = 4356$.

This method furnishes a solution to the following question:—

Find a triangle such that the sides, the radii of the inscribed, circumscribed, and escribed circles, and also the radii of Malfatti's circles, shall be integral numbers.

For, since $R = \frac{1}{2}a \operatorname{cosec} A$, $r_1 = s \tan \frac{1}{2}A$, $r_2 = s \tan \frac{1}{2}B$, $r_3 = s \tan \frac{1}{2}C$, we have, when $r = 13860$,

$$a = 56784,\ b = 42000,\ c = 51744;\ R = 29575,$$

$$r_1 = 56448,\ r_2 = 31360,\ r_3 = 44352,\ \rho_1 = 7938,\ \rho_2 = 9800,\ \rho_3 = 8612.$$

4316. (Proposed by Dr. Hart.)—Find n numbers such that, if each number be subtracted from the cube of their sum, the results shall be cubes.

Solution by Samuel Bills.

Let $x_1, x_2, x_3, \ldots x_n$ denote the n numbers, and s their sum; and suppose

$$s^3-x_1 = s^3a_1^3,\quad s^3-x_2^3 = s^3a_2^3,\quad \ldots\ldots,\quad s^3-x_n = s^3a_n^3;$$

then $\quad x_1 = s^3(1-a_1^3),\quad x_2 = s^3(1-a_2^3),\quad \ldots\ldots,\quad x_n = s^3(1-a_n^3).$

Adding these results, we obtain

$$s = s^3(n-a_1^3-a_2^3 \ldots - a_n^3);\quad \text{whence}\quad s^2 = (n-a_1^3-a_2^3-a_3^3 \ldots - a_n^3)^{-1},$$

and we must have $\quad n-a_1^3-a_2^3 \ldots - a_n^3 = \square = k^2$ (suppose),

then $\qquad a_1^3+a_2^3+a_3^3 = n-k^2-a_4^3-a_5^3 \ldots - a_n^3.$

Now, n being given, let $k, a_4, a_5, \ldots a_n$ be taken at pleasure, and let the right hand side of the above equation be denoted by m; then

$$a_1^3+a_2^3+a_3^3 = m,$$

a given number; and since *any* given number may be resolved into three rational cube numbers, we shall thus obtain a solution of the question.

For a numerical example, take $n=6$, and let $a_4=\frac{1}{2}$, $a_5=\frac{2}{3}$, $a_6=\frac{5}{6}$; also take $k=2$; then $m=1$, and we must have $a_1^3+a_2^3+a_3^3 = 1$. One solution of this is $a_1=\frac{1}{3}$, $a_2=\frac{26}{28}$, $a_3=\frac{53}{84}$. From these we find $s=\frac{1}{4}$; whence we obtain $\quad x_1 = \frac{1}{8}\left\{1-(\frac{1}{3})^3\right\},\quad x_2 = \frac{1}{8}\left\{1-(\frac{26}{28})^3\right\},\quad x_3 = \frac{1}{8}\left\{1-(\frac{53}{84})^3\right\},$

$$x_4 = \frac{1}{8}\left\{1-(\frac{1}{2})^3\right\},\quad x_5 = \frac{1}{8}\left\{1-(\frac{2}{3})^3\right\},\quad x_6 = \frac{1}{8}\left\{1-(\frac{5}{6})^3\right\}.$$

Another solution may be obtained by taking $a_1 = \frac{4}{11}$, $a_2 = \frac{33}{44}$, $a_3 = \frac{41}{44}$.

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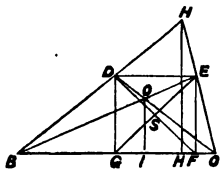
4467. (Proposed by J. JAMESON.)—DEFG is a square inscribed in a triangle ABC whose base BC is given in magnitude and position; BE, CD meet in O; and GE, FD meet in S; show that the straight line OS always passes through the vertex of a semicircle on BC as diameter.

I. *Solution by the* Rev. J. R. WILSON, M.A.; R. TUCKER, M.A.; *and others.*



Let (x', y') be the coordinates of A referred to B as origin, and BC as axis of x; and (x_1, y_1) the coordinates of E. Draw AH perpendicular to BC. Let $BC = a$. Then, to find the coordinates of E, we have, from the triangles AHC, EFC, and AHB, DGB,

$$\frac{y'}{y_1} = \frac{a-x'}{a-x_1} = \frac{x'}{x_1-y_1};$$

whence we obtain $x_1 = \dfrac{a(x'+y')}{a+y'}$, $y_1 = \dfrac{ay'}{a+y'}$.

Hence the equation to BE is $\dfrac{y}{x} = \dfrac{y'}{x'+y'}$ .....(1).

The coordinates of D are easily found to be $\dfrac{ax'}{a+y'}$, $\dfrac{ay'}{a+y'}$.

Thus the equation to CD is $\left(\dfrac{x}{a} + \dfrac{a+y'-x'}{ay'}\right) . y = 1$ ..... (2).

(1) and (2) give, for the coordinates of O, $\dfrac{a(x'+y')}{a+2y'}$, $\dfrac{ay'}{a+2y'}$.

The coordinates of S are $\dfrac{2ax'+ay'}{2(a+y')}$, $\dfrac{ay'}{2(a+y')}$.

Hence the equation to OS is found to be $y(a-2x') = a(x-x')$ hence OS always passes through the point specified in the Question.

The foregoing analytic solution may be improved thus:—
Put (x, y) for the coordinates of O (BI, IO say), and $DE = DG = s$;

then $\dfrac{a}{s} = \dfrac{y'}{y'-s}$, whence $s = \dfrac{ay'}{a+y'}$;

$$\frac{x}{y} = \frac{BI}{IO} = \frac{BF}{FE} = \frac{BG+GF}{FE} = 1 + \frac{x'}{y'} \text{ ..... (3),}$$

$$\frac{a-x}{y} = \frac{IC}{IO} = \frac{GC}{GD} = \frac{GF+FC}{GD} = 1 + \frac{a-x'}{y'} \text{ .....(4);}$$

(3) + (4) gives $\dfrac{a}{y} = 2 + \dfrac{a}{y'}$, therefore $y = \dfrac{ay'}{a+2y'}$; and thence we obtain

$$x = \frac{y}{y'}(x'+y') = \frac{a(x'+y')}{a+2y'};$$

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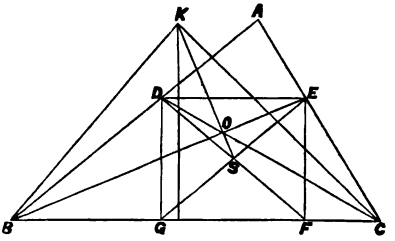
hence the coordinates of the point S, the centre of the square, are

$$y = \tfrac{1}{2}s = \frac{ay'}{2(a+y')} \text{ and } x = \tfrac{1}{2}s + \text{BG} = \tfrac{1}{2}s + \frac{sx'}{y'} = \frac{s}{2}\left(1 + \frac{2x'}{y'}\right) = \frac{a(y'+2x')}{2(a+y')};$$

therefore the equation of OS is $y(a-2x') = a(x-x')$, which is satisfied by $x=y=\frac{1}{3}a$, that is, by the middle point of the semicircle on BC.]

II. *Solution by* G. S. Carr; C. Leudesdorf; *and others.*



In the figure, produce SO to K, making OK a fourth proportional to OE, OB, OS; and join BK and CK. Then, because OE : OB = OS : OK, BK is parallel to SE; also

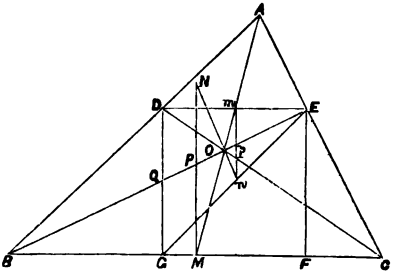
OE : OB = OD : OC,

$\therefore$ OD : OC = OS : OK,

$\therefore$ CK is parallel to SD;

therefore, since BK and CK are parallel to the diagonals of the square, K must be the vertex of the semicircle upon BC.

III. *Solution by* M. Collins, LL.D.; E. B. Elliott, B.A.; *and others.*



Let DEFG be a square in the triangle ABC, O the intersection of BE and CD; then AO plainly bisects BC and DE in M and m; also the middle point n of EG is plainly the centre of the square; hence if nO produced meet in a line MPN drawn through M parallel to GD or Mn, we have four lines diverging from O, and cutting the two parallel lines MPN and mpn; therefore

$$\text{MN} : \text{MP} = mn : mp = m\text{E} : mp = \text{MB} : \text{MP};$$

therefore MN = MB = MC, and N is the vertex of a semicircle on BC.

IV. *Solution by* Professor Townsend, F.R.S.

The two points A and O being the two centres of similitude, external and internal, of the two parallel straight lines BC and DE, all pairs of homologous points of any two similar figures, similarly or oppositely described on those lines, connect, for similar description through A, and for opposite description through O; but the point S is the centre of the square DEFG described as in the figure on DE, and the two vertices S_1

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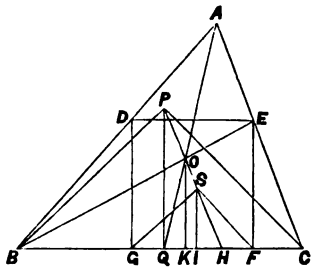
and S_2 of the two semicircles on BC as diameter are those of the two squares on BC as base, of which one is similarly and the other oppositely situated with respect to that square, and therefore, &c.

V. *Solution by* E. RUTTER; J. WILSON; S. TEBAY, B.A.; Rev. T. J. SANDERSON, M.A.; *and others.*

Let ABC be the triangle; DEFG the inscribed square; O and S the points in the question. Draw BP, CP parallel to GS, FS, to meet at P; then the triangles BPC, GSF are right-angled and isosceles; also because DE is parallel to BC, AO produced bisects BC at Q. Hence PQ = QC = BQ; and PQ is perpendicular to BC, because BPC is a right-angled isosceles triangle.



Draw OK, SI perpendicular to BC; join PQ, and let OS meet BC at H. Then, by similar triangles,

$$HI : IS = HK : KO = HQ : QP$$

therefore SO passes through P.

[COR. 1.—By a precisely similar proof, it may be shewn that, if O, S be corresponding points for squares *escribed* to the triangle ABC, OS passes through the other end T of the diameter PT.

COR. 2.—So also, if rhombi of given species be inscribed in, or escribed to, the triangle ABC, and O, S be the points which are the analogues of O, S above, it may be shewn that, for the inscribed figures, OS passes through one end of that diameter which is parallel to the sides of the rhombi; and for the escribed figures, through the other end.

COR. 3.—By orthogonal projection on a plane through BC, we at once obtain the following generalized form of the theorem. If in a triangle ABC, whose base BC is given, there be inscribed a parallelogram DEFG, with a given angle DGF, and such that the diagonal GE makes a constant angle with BC; and if BE, CD meet in O, and GE, FD in S; then OS always passes through one end of the diameter (parallel to DG, EF) of a certain ellipse described on BC as minor axis; and for the corresponding escribed parallelograms, OS passes through the other end.]

2755. (Proposed by WILLIAM LEA.)—Form sixteen symbols into sets, four symbols in each set, so that every triad in the symbols shall appear once in the sets.

Solution by the PROPOSER.

Number of sets $= \frac{16 \times 15 \times 14}{4 \times 3 \times 2} = 140$;

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number of times each symbol appears $= \frac{15 \times 14}{3 \times 2} = 35$;

number of times each pair of symbols appears $= \frac{16-2}{4-2} = 7$;

Let the 16 symbols be as follows:—

A, B, C, D; V, X, Y, Z; v, x, y, z; a_1, a_2, a_3, a_4.

Then $4 \times 16 = 64$ sets are found by the following Table, and the remaining $36 + 24 + 12 + 4 = 76$ sets are, from the combinations, self-evident.

	V	X	Y	Z	z	y	x	v	a_4	a_3	a_2	a_1	4
	V	X	Y	Z	y	z	v	x	a_3	a_4	a_1	a_2	3
	V	X	Y	Z	x	v	z	y	a_2	a_1	a_4	a_3	2
	V	X	Y	Z	v	x	y	z	a_1	a_2	a_3	a_4	1
A	1	2	3	4	1	2	3	4	1	3	4	2	1
B	5	6	7	8	6	5	8	7	8	6	5	7	2
C	9	10	11	12	11	12	9	10	10	12	11	9	3
D	13	14	15	16	16	15	14	13	15	13	14	16	4

64 SETS.

AV	va_1	AX	xa_4	AY	ya_2	AZ	za_3
,,	xa_2	,,	va_3	,,	za_1	,,	ya_4
,,	ya_3	,,	za_2	,,	va_4	,,	xa_1
,,	za_4	,,	ya_1	,,	xa_3	,,	va_4
BV	xa_3	BX	va_2	BY	za_4	BZ	ya_1
,,	va_4	,,	xa_1	,,	ya_3	,,	za_2
,,	za_1	,,	ya_4	,,	xa_2	,,	va_3
,,	ya_2	,,	za_3	,,	va_1	,,	xa_4
CV	ya_4	CX	za_1	CY	va_3	CZ	xa_2
,,	za_3	,,	ya_2	,,	xa_4	,,	va_1
,,	va_2	,,	xa_3	,,	ya_1	,,	za_4
,,	xa_1	,,	va_4	,,	za_2	,,	ya_3
DV	za_2	DX	ya_3	DY	xa_1	DZ	va_4
,,	ya_1	,,	za_4	,,	va_2	,,	xa_3
,,	xa_4	,,	va_1	,,	za_3	,,	ya_2
,,	va_3	,,	xa_2	,,	ya_4	,,	za_1

36 SETS.

AB	VX	AC	VY	AD	VZ	BC	VZ	BD	VY	CD	VX
,,	YZ	,,	XZ	,,	XY	,,	XY	,,	XZ	,,	YZ
,,	vx	,,	vy	,,	vz	,,	vz	,,	vy	,,	vx
,,	yz	,,	xz	,,	xy	,,	xy	,,	xz	,,	yz
,,	a_1a_2	,,	a_1a_3	,,	a_1a_4	,,	a_1a_4	,,	a_1a_3	,,	a_1a_2
,,	a_3a_4	,,	a_2a_4	,,	a_2a_3	,,	a_2a_3	,,	a_2a_4	,,	a_3a_4

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24 SETS.

VX vx	VY vy	VZ vz	XY vz	XZ vy	YZ vx
,, yz	,, xz	,, xy	,, xy	,, xz	,, yz
,, a_1a_2	,, a_1a_3	,, a_1a_4	,, a_1a_4	,, a_1a_3	,, a_1a_2
,, a_3a_4	,, a_2a_4	,, a_2a_3	,, a_2a_3	,, a_2a_4	,, a_3a_4

12 SETS.

vx a_1a_2	vy a_1a_3	vz a_1a_4	xy a_1a_4	xz a_1a_3	yz a_1a_2
,, a_3a_4	,, a_2a_4	,, a_2a_3	,, a_2a_3	,, a_2a_4	,, a_3a_4

4 SETS.

ABCD	VXYZ	$vxyz$	$a_1a_2a_3a_4$

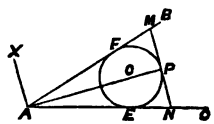
Sum of all the Sets = 64 + 36 + 24 + 12 + 4 = 140.

[See also *Reprint*, Vol. XI., p. 97.]

4192. (Proposed by T. T. Wilkinson, F.R.A.S.)—Draw a circle to touch two straight lines given in position, and such that, if a tangent be drawn to it parallel to a third line given in position terminating in the lines, the rectangle of the segments made at the point of contact shall be equal to a given space.

Solution by the Proposer.

Analysis.—Suppose the problem solved; let AB, AC be the lines given in position; (O) the circle touching them in F and E; MN the line parallel to AX given in position, touching the circle in P. Then, because MN is parallel to AX, it makes *given* angles with AB, AC. Consequently, the triangle AMN is given in species. But MP : PN = a given ratio; also MP : PN = a given rectangle. Hence by "lineal sections," both MP and PN may be determined. But MN is a given line, and therefore the triangle AMN is wholly given. The problem is consequently reduced to that of inscribing a circle in a given triangle.



4519. (Proposed by A. Martin.)—Find the equation to the envelope of equal chords of a given ellipse.

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Solution by the EDITOR.

1. Let (x, y) be a point on the given ellipse, through which is drawn a secant whose tangential coordinates are (ξ, v); then we shall have

$$a^2y^2 + b^2x^2 = a^2b^2, \quad \xi x + vy = 1 \ldots\ldots\ldots\ldots (1, 2).$$

Eliminating y, we have $x^2 - \frac{2a^2\xi \cdot x}{a^2\xi^2 + b^2v^2} + \frac{a^2(1-b^2v^2)}{a^2\xi^2 + b^2v^2} = 0 \ldots\ldots\ldots\ldots (3).$

Let (x_1, x_2) be the roots of (3), that is to say, the abscissas of the ends of the chord; then we shall have

$$x_1 + x_2 = \frac{2a^2\xi}{a^2\xi^2 + b^2v^2}, \text{ and } x_1x_2 = \frac{a^2(1-b^2v^2)}{a^2\xi^2 + b^2v^2} \ldots\ldots\ldots\ldots (4),$$

whence $$(x_1 - x_2)^2 = \frac{4a^2b^2v^2(a^2\xi^2 + b^2v^2 - 1)}{(a^2\xi^2 + b^2v^2)^2} \ldots\ldots\ldots\ldots (5);$$

similarly $(y_1 - y_2)^2 = \&c.$

Hence, if $2c$ be the length of the constant chord, so that

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 4c^2,$$

we obtain, by substitution from (5), the following tangential equation of the required envelope, $a^2b^2(\xi^2 + v^2)(a^2\xi^2 + b^2v^2 - 1) = c^2(a^2\xi^2 + b^2v)^2 \ldots\ldots (6).$

2. Let P be the perpendicular from the centre on the tangent to this curve, and λ the angle which P makes with the major axis; then (6) may be transformed into

$$a^2b^2(a^2\cos^2\lambda + b^2\sin^2\lambda - P^2) = c^2(a^2\cos^2\lambda + b^2\sin^2\lambda)^2,$$

or $$a^2b^2P^2 = (a^2\cos^2\lambda + b^2\sin^2\lambda)\left\{(b^2 - c^2)a^2\cos^2\lambda + (a^2 - c^2)b^2\sin^2\lambda\right\} \ldots (7).$$

3. Let p be the perpendicular on the tangent to the ellipse coinciding with P, then $$p^2 = a^2\cos^2\lambda + b^2\sin\lambda,$$

therefore $$a^2b^2(p^2 - P^2) = c^2p^4, \text{ or } \left(\frac{P}{p}\right)^2 = 1 - \left(\frac{cp}{ab}\right)^2 \ldots\ldots\ldots (8).$$

4. We may find the projective equation of the locus of the foot of a perpendicular from the centre on the moving constant chord, that is to say, the central *pedal* of the envelope, by substituting in the tangential equation (6) $\frac{x}{x^2 + y^2}$ for ξ and $\frac{y}{x^2 + y^2}$ for v (see Dr. BOOTH's *New Geometrical Methods*, p. 159); and we thus obtain

$$a^2b^2(x^2 + y^2)\left\{a^2x^2 + b^2y^2 - (x^2 + y^2)^2\right\} = c^2(a^2x^2 + b^2y^2)^2 \ldots\ldots\ldots (9).$$

5. When $c = 0$, the tangential equation of the envelope (6) becomes

$$a^2\xi^2 + b^2v^2 - 1 = 0,$$

that is, the tangential equation of the given ellipse, as it ought to be.

When $c = 0$, (9) becomes $a^2x^2 + b^2y^2 = (x^2 + y^2)^2$, as it ought to do.

4523. (Proposed by J. C. W. ELLIS, M.A.)—A pavement is formed of equal elliptical slabs of white marble, their major axes being each $2a$ and pointing to the north, and minor axes $2b$. Any four contiguous slabs

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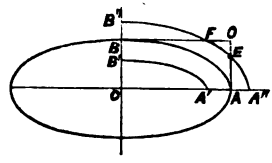
have their centres at the angles of an oblong whose sides are $2a$ and $2b$. The interstices are filled up with black marble. A black elliptical lamina, whose axes are $2ma$, $2mb$, and a white one, whose axes are $2na$, $2nb$, are dropped at random on the pavement, and take up positions with their major axes pointing to the north. Find the chance of the black one being entirely on a white surface, and the white one on a black. If $a = b$, and the radius of a silver coin be $\frac{1}{8}(2\sqrt{3}-3)\,a$, and of a copper one $\frac{1}{10}a$; show that the chance of the silver lying wholly on the black, and the copper wholly on the white, is $\frac{8}{1000}\pi\,(9-\pi-3\sqrt{3})$.

Solution by the Rev. T. J. SANDERSON, M.A.

We observe first that the laminæ are both similar to the elliptical slabs, and that they fall so as to be similarly placed.



Hence the locus A′B′ of the centre of the black lamina, and the locus A″B″ of the centre of the white lamina, when the one is just within and the other just without a given slab AB, will be similar ellipses, of which the semi-axes will be $a(1-m)$, $b(1-m)$, and $a(1+n)$, $b(1+n)$.

Now the chance that the black lamina should fall on a white slab is area A′B′C : rect. CO $= \frac{1}{4}\pi(1-m)^2$, and the chance that the white lamina should fall on a black interspace is area EFO : rect. CO.

Now area EFO = rect. CO − area A″B″C + area FBB″ + area EAA″;

and by a simple integration we may find that

$$\text{area FBB}'' = \tfrac{1}{4}ab\left\{\tfrac{1}{4}\pi(n+1)^2-(n^2+2n)^{\frac{1}{2}}-(n+1)^2\operatorname{cosec}^{-1}(n+1)\right\} = \text{EAA}'';$$

$$\therefore\ \text{area EFO} = ab\left\{1+\tfrac{1}{4}\pi(n+1)^2-(n^2+2n)^{\frac{1}{2}}-(n+1)^2\operatorname{cosec}^{-1}(n+1)\right\}.$$

Hence the required chance of the simultaneous occurrence of the two events is $\frac{1}{4}\pi(1-m)^2\left\{1+\frac{1}{4}\pi(n+1)^2-(n^2+2n)^{\frac{1}{2}}-(n+1)^2\operatorname{cosec}^{-1}(n+1)\right\}$.

In the special case given, we have $m=\frac{1}{10}$, $n=\frac{4}{5}\sqrt{3}-1$; and the required chance then becomes that given in the Question.

4520. (Proposed by A. B. EVANS, M.A.)—Find the least integral values of x and y that will satisfy the equation $x^2 - 953y^2 = -1$.

I. *Solution by* Professor CAYLEY.

The values are given in Degen's Tables, viz.,

$$x = 2746864744,\quad y = 88979677.$$

The work referred to is entitled "Canon Pellianus, sive Tabula simplicissimam æquationis celebratissimæ $y^2 = ax^2+1$ solutionem pro singulis numeri dati valoribus ab 1 usque ad 1000 in numeris rationalibus iisdemque integris exhibens. Autore C. F. Degen, Hafniæ (Copenhagen), 1817."

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Table I., pp. 3—106 gives, for all numbers 1 to 1000, the denominators, and (?) quotients of the convergent fraction of $\sqrt{a}$, also the least values of x, y which will satisfy the equation $x^2 - ay^2 = +1$. Thus

953	30, 1, 6, 1, 2, 1, 3, 8, 1, 1, (4, 4)	
	1, 53, 8, 41, 17, 37, 16, 7, 32, 29, (13, 13)	
	488830275367615376,	150905318436603710 73

Table II., pp. 109—112, is described as giving for all those values of a between 1 and 1000, for which there exists a solution of the equation $x^2 - ay^2 = -1$, the least values of x and y which satisfy this equation: thus 953, x and y as above.

It is, however, to be noticed that the values of $a = \beta^2 + 1$, for which there is the obvious solution $x = \beta$, $y = 1$, are omitted from the table. The reason for this appears, but the heading should have been different.

II. *Solution by* M. Collins, LL.D.; R. Tucker, M.A.; *and others.*

Since x and y are required to be integers in the given equation

$$x^2 - 953y^2 = -1 \quad \text{or} \quad \frac{x^2}{y^2} = 953 - \frac{1}{y^2},$$

and since no rational values of x and y can make $x^2 - 953y^2 = 0$, therefore $\frac{x}{y}$ must be as near as possible to $(953)^{\frac{1}{2}}$ so as to render $x^2 - 953y^2$ a minimum integer; and as its value (-1) is negative, therefore $\frac{x}{y}$ must be $<$, and not $>$ $(953)^{\frac{1}{2}}$. Moreover, as $\frac{x^2+1}{953}$ = the integer y^2, and this would be impossible (as proved in Art. 14 of my tract "*On the possible and impossible cases of quadratic duplicate equalities in the Diophantine Analysis*," or in Art. 105 of Barlow's *Theory of Numbers*) unless 953 was the sum of two squares, and as $953 = 28^2 + 13^2$, therefore the question may be possible, and if so $\frac{x}{y}$ must be one of the principal fractions converging to and less than $(953)^{\frac{1}{2}}$ (by Art. 38 of Lagrange's *Additions to Euler's Algebra*); and if the periodic continued fraction for $(953)^{\frac{1}{2}} = N^{\frac{1}{2}}$ has a period consisting of an odd number of terms, then the question is possible, and not otherwise; and so the present question is possible, as the said period consists of 21 terms, found as follows by the usual method, only adopting for the sake of brevity and expedition the known principle that if $\frac{N^{\frac{1}{2}} + m}{n}$ be one of the *complete quotients*, the next following one $\frac{N^{\frac{1}{2}} + m'}{n'}$ will be such that $m' = nu - m$ and $nn' = N - m'^2$, where u is the greatest whole number in $\frac{N^{\frac{1}{2}} + m}{n}$. Now we begin with $(953)^{\frac{1}{2}}$ or $\frac{(953)^{\frac{1}{2}} + 0}{1} = 30 = a$, and the following complete quotients are

$$\frac{(953)^{\frac{1}{2}}+30}{53} = 1, \quad \frac{(953)^{\frac{1}{2}}+23}{8} = 6, \quad \frac{(953)^{\frac{1}{2}}+25}{41} = 1, \quad \frac{(953)^{\frac{1}{2}}+16}{17} = 2,$$

$$\frac{(953)^{\frac{1}{2}}+18}{37} = 1, \quad \frac{(953)^{\frac{1}{2}}+19}{16} = 3, \quad \frac{(953)^{\frac{1}{2}}+29}{7} = 8, \quad \frac{(953)^{\frac{1}{2}}+27}{32} = 1,$$

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$$\frac{(953)^{\frac{1}{2}}+5}{29}=1, \quad \frac{(953)^{\frac{1}{2}}+24}{13}=4, \quad \frac{(953)^{\frac{1}{2}}+28}{13}=4, \quad \frac{(953)^{\frac{1}{2}}+24}{29}=1,$$

$$\frac{(953)^{\frac{1}{2}}+5}{32}=1, \text{ \&c. \&c. up to } \frac{(953)^{\frac{1}{2}}+23}{53}=1,$$

and $$\frac{(953)^{\frac{1}{2}}+30}{1}=60=2a=\frac{(953)^{\frac{1}{2}}+a}{1},$$

which are the two last terms of the period of complete quotients that begins with the second complete quotient $\frac{(953)^{\frac{1}{2}}+30}{53}$, for the first complete quotient $\frac{(953)^{\frac{1}{2}}+0}{1}=a$ never recurs again at all.

Thus the series of integers next < the series of complete quotients are 30, 1, 6, 1, 2, 1, 3, 8, 1, 1, 4, 4, 1, 1, 8, 3, 1, 2, 1, 6, 1, 60.

Hence the continued fraction for $(953)^{\frac{1}{2}}$ is

$$30+\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{8+}\text{ \&c.;}$$

and the series of fractions converging to $(953)^{\frac{1}{2}}$ are

$$\frac{30}{1}, \frac{31}{1}, \frac{216}{7}, \frac{247}{8}, \frac{710}{23}, \frac{957}{31}, \left(\frac{3581}{116}\right), \frac{29605}{959}, \frac{33186}{1075},$$
$$\frac{62791}{2034}, \frac{284350}{9211}, \frac{1200191}{38878}, \frac{1484541}{48089}, \frac{2684732}{86967},$$
$$\frac{22962397}{743825}, \frac{71571923}{2318442}, \frac{94534320}{3062267}, \frac{260640563}{8442976},$$
$$\frac{355174883}{11505243}, \frac{2391689861}{77474434}, \frac{2746864744}{88979677}.$$

The 21st term of this series,—which just precedes the 22nd or last complete quotient $\frac{(953)^{\frac{1}{2}}+a}{1}=2a$ of the period whose denominator is always $=1$,—is $\frac{2746864744}{88979677}=\frac{x}{y}$, as required.

If the 7th convergent $\frac{3581}{116}$ be taken for $\frac{x}{y}$, then $x^2-953y^2=-7$, which is the denominator of the next following or 8th complete quotient $\frac{(953)^{\frac{1}{2}}+29}{7}$ (see Art. 639 of Todhunter's *Algebra*), and it is plain that the foregoing solution thus furnishes also solutions of

$$x^2-953y^2=\pm c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)},$$

when c is any of the numbers

$$1, 7, 8, 13, 16, 17, 29, 32, 37, 41, 53 \quad \ldots\ldots\ldots\ldots \text{(B)},$$

and so the equation (A) is, most probably, impossible in rational numbers when c is an integer < 53 and not contained in the series (B), which consists of the denominators of the complete quotients for $(953)^{\frac{1}{2}}$. For a method of finding other solutions of equation (A) from one known solution, see Todhunter's *Algebra*, Art. 643, 644.

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4508. (Proposed by Professor WOLSTENHOLME, M.A.)—A large number of equal particles are fastened at unequal intervals to a fine string, and then collected into a heap at the edge of a smooth horizontal table with the extreme one just hanging over the edge; the intervals are such that the times between successive particles being carried over the edge are equal; prove that if c_n be the interval between the nth and $(n+1)$th particle, and v_n the velocity just after the $(n+1)$th particle is carried over, then

$$c_1^{-1}c_n = v_1^{-1}v_n = n.$$

I. *Solution by* Professor TOWNSEND, M.A., F.R.S.

Denoting by v the velocity acquired during the fall of the first particle through the interval c_1 between it and the second, we have immediately, from the conditions of the question, the two series of relations,

$$v_1 = \tfrac{1}{2}v, \quad v_2 = \tfrac{2}{3}(v_1+v) = \tfrac{2}{3}v, \quad v_3 = \tfrac{3}{4}(v_2+v) = \tfrac{3}{2}v, \text{ \&c.},$$

$$2g\,.\,c_1 = v^2, \quad 2g\,.\,c_2 = (v_1+v)^2 - v_1^2 = 2v^2, \quad 2g\,.\,c_3 = (v_2+v)^2 - v_2^2 = 3v^2, \text{ \&c.},$$

from which it follows at once that

$$v_1 : v_2 : v_3 : \text{\&c.} : v_n = c_1 : c_2 : c_3 : \text{\&c.} : c_n = 1 : 2 : 3 : \text{\&c.} : n,$$

and therefore, &c.

II. *Solution by the* Rev. T. J. SANDERSON, M.A.; G. S. CARR; *and others.*

As each particle in turn is added to the descending mass, the velocity varies inversely as that mass. Hence we have the equations

$$v_1 = \tfrac{1}{2}gt, \quad v_2 = \tfrac{2}{3}(v_1+gt) = 2v_1, \quad v_3 = \tfrac{3}{4}(v_2+gt) = 3v_1 \ \ldots\ldots, \ \ldots\ldots (1),$$

$$v_n = \frac{n}{n+1}(v_{n-1}+gt) = nv_1 \ \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also $$c_n = v_{n-1}t + \tfrac{1}{2}gt^2 = (n-1)v_1t + v_1t, \text{ by (2) and (1)},$$

$$= nv_1t = nc_1, \text{ because } c_1 = \tfrac{1}{2}gt^2.$$

Thus $$v_1^{-1}v_n = c_1^{-1}c_n = n.$$

III. *Solution by* S. TEBAY, B.A.

Let v_n be the velocity just before the $(n+1)$th particle is carried over, and w the weight of each. Then we have

$$v_n^2 = v_{n-1}^2 + 2gc_n, \quad v_n = v_{n-1} + 2v_1 = v_{n-1} + (2gc_1)^{\frac{1}{2}},$$

$$v_n = \frac{nw(v_n)}{nw+w} = \frac{n}{n+1}(v_{n-1}+v_1);$$

therefore $$v_n - \frac{n}{n+1}v_{n-1} = \frac{2nv_1}{n+1}.$$

Let $v_{n-1} = \frac{u_n}{n}$; then we have $\Delta u_n = 2nv_1$,

$$u_n = 2v_1\left\{\tfrac{1}{3}n(n-1)+c\right\}, \quad v_1^{-1}v_{n-1} = \frac{2}{n}\left\{\tfrac{1}{3}n(n-1)+c\right\}, \quad 1 = 1+c;$$

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and therefore $v_1^{-1} v_n = n.$

Hence $(v_n^2) = \frac{1}{2} g c_1 (n-1)^2 + 2g c_n = \left\{(n-1)(\frac{1}{2} g c_1)^{\frac{1}{2}} + (2g c_1)^{\frac{1}{2}}\right\}^2,$

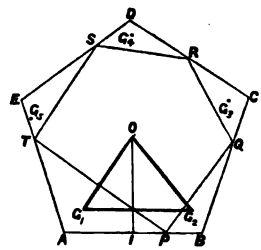
and $c_1^{-1} c = n.$

3413. (Proposed by S. WATSON.)—On the sides of a regular pentagon five points are taken at random, one on each, which when joined will form an inscribed pentagon. Again, in the five triangles outside this pentagon, five other points are taken at random, one in each, forming when joined another pentagon, the average area of which is required.

3661. (Proposed by S. WATSON.)—On the sides of a regular polygon of n sides, n points are taken at random, one on each side, forming when joined an inscribed polygon of n sides. Again, in the n triangles outside this last polygon, n points are taken at random, one in each, forming when joined a third polygon of n sides, the average area of which is required.

Solution by the PROPOSER.

3413. Let ABCDE be the pentagon; O its centre; P, Q, R, S, T the first five points; G_1, G_2 ... G_5 the centres of gravity of the triangles TAP, PBQ ... SET; and I the middle of AB. Put



$$AB = 2a, \quad IP = x, \quad BQ = y, \quad CR = y_1,$$
$$AT = z, \quad ES = z_1, \quad \angle ABC = \alpha.$$

Then, if y_1, y_2, y_3 be the distances of O, G_1, G_2 from AB, and x_2, x_3 of G_1, G_2 from OI, we have

$$y_1 = a \tan \tfrac{1}{2}\alpha, \quad y_2 = \tfrac{1}{3} z \sin \alpha, \quad y_3 = \tfrac{1}{3} y \sin \alpha,$$
$$x_2 = \tfrac{1}{3}(2a - x - z\cos\alpha), \quad x_3 = \tfrac{1}{3}(2a + x - y\cos\alpha);$$

hence the area of the triangle OG_1G_2 is

$$\tfrac{1}{2}\left\{(y_1+y_2)x_2 + (y_1+y_3)x_3 - (y_2+y_3)(x_2+x_3)\right\}$$
$$= \tfrac{1}{2}\left\{(y_1-y_3)x_2 + (y_1-y_2)x_3\right\}$$
$$= a\tan\tfrac{1}{2}\alpha\left\{\tfrac{2}{3}a - \tfrac{1}{6}(y+z)\cos\alpha\right\} + \tfrac{1}{6}\sin\alpha\cos\alpha\, yz$$
$$-\tfrac{1}{18}\sin\alpha\left\{(2a+x)z + (2a-x)y\right\} \ldots\ldots\ldots (1).$$

Now when the second five points take all positions in their respective triangles, the average area of the pentagon formed by joining them is the pentagon $G_1G_2 \ldots G_5$, and the number of pentagons is

$$\frac{1}{2^5}\sin^5\alpha\,(a-x)\,y\,(a+x)\,z\,(2a-y)\,y_1\,(2a-z)\,z_1\,(2a-z_1) \ldots\ldots\ldots (2).$$

Multiplying by 5 because each of the triangles OG_1G_2, OG_2G_3, &c., will

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pass through the same magnitudes, the required average is

$$\frac{5\int_{-a}^{a}dx\int_{0}^{2a}dy\int_{0}^{2a}dz\int_{0}^{2a}dy_1\int_{0}^{2a}dz_1\,(1)\,(2)}{\int_{-a}^{a}dx\int_{0}^{2a}dy\int_{0}^{2a}dz\int_{0}^{2a}dy_1\int_{0}^{2a}dz_1\,(2)} \quad \ldots\ldots\ldots\ldots (3)$$

$= \frac{5}{8}(8\sin^2\frac{1}{5}a + \cos^2 a)\,a^2\tan\frac{1}{5}a = \frac{1}{8}(8\sin^2\frac{1}{5}a + \cos^2 a)$ of given pentagon.

3661. Since (3) remains the same if we omit the integrations with respect to y_1, z_1 it is easily seen to follow that the final result will be the same for a regular polygon of n sides, a being $= \frac{(n-2)\pi}{n}$; hence the required average in this Question is

$$\frac{1}{9}\left\{8\sin^2\frac{(n-2)\pi}{2n} + \cos^2\frac{(n-2)\pi}{n}\right\} \text{ of given polygon}$$

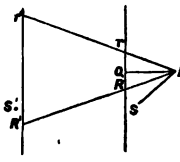
$$= \frac{n}{9}\left\{8\sin^2\frac{(n-2)\pi}{2n} + \cos^2\frac{(n-2)\pi}{n}\right\} a^2\tan\frac{(n-2)\pi}{2n}.$$

[Making $n = 3$ and 4, we have respectively $\frac{1}{4}$ of given triangle and $\frac{4}{9}$ of given square for the averages required in Questions 2791, 3188.]

4465. (Proposed by T. Cotterill, M.A.)—If a circle has its centre on a conic and passes through a focus, prove that the diameters of the circle through its points of intersection with the directrix of the focus are parallel to the asymptotes of the conic.

Solution by the Rev. J. R. Wilson, M.A.

1. Let P be a point on the conic. With centre P draw a circle to pass through the focus S; and let this circle cut the corresponding directrix in R, r. Draw PQ perpendicular to Rr. Then if e denote the eccentricity of the conic,



$$RP = SP = e\,.\,PQ, \text{ or } \cos QPR = e^{-1}.$$

But e^{-1} is the cosine of the angle which the asymptote makes with the axis of the conic. Hence PR is parallel to an asymptote. In like manner, Pr is parallel to the other asymptote.

2. If S′ be the other focus, and R′, r' points on the other directrix, corresponding to the points R, r; then PRR′, Prr' are straight lines; and hence, if from any point on a conic a straight line be drawn parallel to an asymptote, the part intercepted by the directrices is equal to the sum or difference of the focal distances of the point.

3. We have also $PR\,.\,PR' = Pr\,.\,Pr' = SP\,.\,S'P$.

4. If a circle be described upon RR′ as diameter, the tangent to it from P will be equal to the conjugate semi-diameter. For $PR\,.\,PR' = SP\,.\,S'P$ = square on the conjugate semi-diameter.

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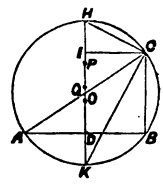
[The theorem in the Question follows at once from the theorem, that if of a conic, a point, the eccentricity, and a directrix are given, then the corresponding focus describes a circle with the given point as centre; also the circles on the major axis and the latus rectum as diameters are circles of the nature described in the Question. Mr. TUCKER proves the theorem thus:—The equation to the circle is $(x-x')^2+(y-y')^2=(ex'-a)^2$, and this meets the directrix in the points $(y-y')^2=(ex'-a)^2\frac{b^2}{a^2}$; therefore $y=y'\pm\frac{b}{a}(ex'-a)$, therefore, &c.]

4210. (Proposed by T. T. WILKINSON, F.R.A.S.)—Having given the difference of the segments of the base made by the perpendicular, and the difference of the angles at the base; to construct the triangle when the sum of the squares on the perpendicular and the base is either given or a maximum.

Solution by the PROPOSER.



Analysis.—Let ABC be the required triangle; HK the diameter of the circumscribing circle bisecting AB in D; O its centre; also join CH, CK, and draw CI parallel to AB. Then CI = half the difference of the segments of base; and CKH = half the difference of the angles at base; also IK, HK, OI are given lines. Now $DI^2+AB^2=(OI+OD)^2+4(BO^2-DO^2)$ is given, or a maximum by the question. Take from both sides of this expression the known quantity OI^2+4BO^2, and a little reduction gives $\frac{3}{2}OI\,.\,DO-DO^2=$ a given space, or a maximum. Make $OQ=\frac{3}{2}OI$, and $OP=OD$; then $(OQ-OP)OP=PO\,.\,PQ=$ a given space, or a maximum. Hence, if OQ be divided in P so that $PO\,.\,PQ$ may be given, or a maximum, the problem is solved. But the method of doing this is well known, and may be omitted. Hence, &c.

4484. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—If a parabola is circumscribed by a quadrilateral two of whose sides are fixed, and the other two are variable in position, prove that the latter intercept on the former segments which are always in a constant ratio to each other.

I. *Solution by* A. M. NASH, B.A.

The tangential equation of the parabola referred to the fixed tangents is (Booth's *New Geometrical Methods*, Art. 55),

$$g\xi\upsilon+h\xi+h_1\upsilon=0.$$

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Let (a_1^{-1}, b_1^{-1}), (a_2^{-1}, b_2^{-1}) be the coordinates of the variable tangents; then the intercepts on the fixed tangents are (a_1-a_2), (b_1-b_2); hence, substituting in the equation of the parabola, we have

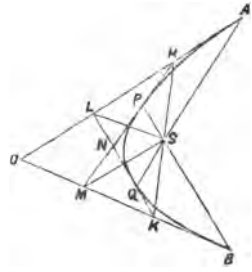
$$g + hb_1 + h_1a_1 = 0, \quad g + hb_2 + h_1a_2 = 0;$$

therefore
$$\frac{a_1-a_2}{b_1-b_2} = -\frac{h}{h_1} = \text{constant}.$$

[Dr. Booth remarks that, as the circumscribing quadrilateral will afford six pairs of axes of coordinates, we may, more generally, infer that, if a quadrilateral be circumscribed to a parabola, any two of its sides will intercept, on the other two, segments which will be in a constant ratio, namely, that of the tangents of the first pair of sides of the quadrilateral.]

II. *Solution by* R. F. Davis, B.A.

Let OA, OB be the fixed, and HPM, LQK the variable tangents to the parabola, forming the quadrilateral OLNM.



Then $\angle HSM = \frac{1}{2}ASB = LSK$,

therefore $\angle HSL = MSK$.

Again $SL^2.SM^2 = SA.SP.SQ.SB = SH^2.SK^2$,

therefore $SL : SH = SK : SM$;

and the triangles HSL, MSK are similar,

therefore $HL^2 : MK^2 = SH^2 : SM^2$
$= SA.SP : SB.SP$
$= SA : SB$
$=$ a constant ratio.

4425. (Proposed by Christine Ladd.) — Find the nature of the *catenaria volvens*, or the figure which a perfectly flexible chain of uniform density and thickness will assume, when it revolves with a constant angular velocity about an axis, to which it is fastened at its extremities, in free and non-gravitating spaces.

[This problem was proposed by Robert Adrian, of Pennsylvania, in No. 6 of the *Mathematical Correspondent*, published in New York about two-thirds of a century ago.]

Solution by G. S. Carr.

Let OA be the semi-axis of revolution, B the vertex of the curve, C any point in the curve. The portion BC of the chain will be in equilibrium under the action of three forces; namely, P the tension at B parallel to AO; T the tension at C in the direction of the tangent to the curve; and the resultant effective force on BC produced by rotation about OA, which last force $= \int y\omega^2 ds$, taking OA for the axis of x and ω for the velocity

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of rotation. Therefore, by the triangle of forces, we have

$$-P\frac{dy}{dx} = \int y\omega^2 ds = \int y\omega^2 \frac{ds}{dx}.dx; \text{ therefore } y\omega^2\frac{ds}{dx} = -P\frac{d^2y}{dx}.$$

Writing m for $\frac{\omega^2}{2P}$, we have $2my\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = -\frac{d^2y}{dx^2}$.

Let $\frac{dy}{dx} = p$, then $\frac{d^2y}{dx^2} = p\frac{dp}{dy}$, thus $2my\,dy = -\frac{p\,dp}{(1+p^2)^{\frac{1}{2}}}$.

Integrating, and taking $y=b$ the greatest ordinate, when $p=0$ we obtain for the differential equation to the curve,

$$\frac{dy}{dx} = \left\{(1+mb^2-my^2)^2-1\right\}^{\frac{1}{2}},$$

a result which agrees with that printed on p. 489 of ROUTH's *Rigid Dynamics*, where the same question is proposed for solution. We have, therefore,

$$x = \int \frac{dy}{\left\{(1+mb^2-my^2)^2-1\right\}^{\frac{1}{2}}} \quad \ldots\ldots (1).$$

Again

$$\int ds = \int \left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} dy,$$

or

$$s = \int \frac{1+mb^2-my^2}{\left\{(1+mb^2-my^2)^2-1\right\}^{\frac{1}{2}}}\,dy \quad \ldots\ldots (2).$$

The integrals of (1) and (2) give x and s respectively in terms of y. Assume $x=f(y)$, $s=\phi(y)$; then we shall have the two equations

$$a = f(0)-f(b) \text{ and } l = \phi(0)-\phi(b) \quad \ldots\ldots (3),$$

where a = the semi-axis OA, and l = half the length of the chain. These equations determine the values of P and b, the two unknown constants involved.

The integrals (1) and (2) are reducible to elliptic functions (HYMER's *Integ. Calc.*, pp. 214, 216). The resulting equations for (3) however, being transcendental and not expressible in finite terms, I am unable to see how the elimination of P and b can be effected, or even an approximation to their values obtained. It would be interesting to know the form of the curve when the velocity ω is vanishing or becoming infinite.

4498. (Proposed by H. MURPHY.)—Given two sides of a triangle, the product of whose base by the square of the perpendicular is a maximum; prove that the product of the tangents of the angles at the base is 2.

Solution by R. F. DAVIS; *the* PROPOSER; *and others.*

Let b, c be the given sides; then $d(p^2a) = 0$, or $p.da+2a.dp = 0$. Differentiating the well known relations $a^2 = b^2+c^2-2bc\cos A$ and $pa = bc\sin A$, we get $a.da = bc\sin A.dA$, and $p.da+a.dp = bc\cos A.dA$. These equations give

$$pda = 2bc.\cos A.dA, \text{ and } a.da = bc.\sin A.dA;$$

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whence $\frac{p}{a} = 2\cot A = -2\cot(B+C) = \frac{2-2\cot B\cot C}{\cot B+\cot C}$;

but $p(\cot B+\cot C) = a$, therefore $\tan B\tan C = 2$.

[*Otherwise.* Put h, k for the parts into which the perpendicular divides the base; then $p^2(h+k)$ is to be a maximum, therefore

$$2(h+k)p - \frac{p^3}{h} - \frac{p^3}{k} = 0,$$

whence $$2 = \frac{p^2}{hk} = \tan B\tan C.]$$

4499. (Proposed by M. COLLINS, LL.D.)—Required a short rule or method for finding the remainder of the division (without the trouble of the long actual work) when a great whole number expressed by (or containing) say a hundred or more arithmetical figures is divided by 73 or 47.

I. *Solution by the* Rev. W. J. GREENFIELD, M.A.

The remainder from a division by 73 is readily obtained by the following rule:—(1) Separate the figures into periods of *four*, beginning from the *right*; (2) Subtract the first period from the second period; (3) Subtract the remainder from the next period, taking care to "carry" from period to period, as in ordinary subtraction; (4) Repeat this operation with all the periods; (5) Divide the last remainder (increased by 1, if required by 3) by 73; the figures left from this division will be the remainder sought, *or its complement*, according as *an even* or *an odd* number of subtractions has been effected.

The *truth* of the rule may be thus proved. Let N be the number; m_1 the value of the four digits on the right; n_1 the value of the remaining digits, *standing alone*; then

$$N = n_1\times 10^4 + m, \quad \text{therefore } N + (n_1 - m_1) = n_1(10^4+1).$$

Now 10^4+1 is *exactly* divisible by 73, therefore the remainders from dividing N and n_1-m_1 must be *complementary*. In like manner, if m_2 be the value of the last four digits in n_1-m_1, and n_2 the value of the others, the remainders from the division by 73 of n_1-m_1 and of n_2-m_2 will be complementary. So that the remainder from the division of N will be the *same* as that from the division of n_2-m_2. Continuing the process here indicated, we arrive at the rule already enunciated.

As an example, the process of finding the remainder from the division by 73 of the number below is as follows:—

91,2673,4586,0912,7843,1055,2890,3490,1461,0475

8266,1826,0847,3739,7174,0669,0386,2504,0986.

Dividing 8266 by 73, the remainder is 17, which is the *complement* of the remainder sought, as *nine* subtractions have been effected; the remainder therefore is 56.

The remainder from dividing by 47 may be found by the same rule, if we substitute periods of 23 *figures* for those of *four* in the rule.

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II. *Solution by the* PROPOSER.

1. Let Mp denote a multiple of p ($=73$), and N the very great whole number, the remainder of whose division by 73 (or 47) is required. Since

$$73 \times 137 = 10^4 + 1, \text{ therefore } 10^4 = -1 + 73 \times 137 = -1 + Mp,$$

whose square and cube &c. give

$$10^8 = 1 + M'p, \quad 10^{12} = -1 + M''p;$$

hence, dividing N from right to left into periods a, b, c, d, &c., each containing four figures,—but the number of figures in the *last* period may be more or less than 4,—then N is plainly

$$= a + b\,(10)^4 + c\,(10)^8 + d\,(10)^{12}, \text{ &c.},$$

and therefore the remainder of $\frac{N}{73}$ or $\frac{N}{p}$ is = the remainder of

$$\frac{a - b + c - d + \text{&c.}}{73}.$$

2. When $p = 47$; since $10^7 + 2 = 141 \times 70922 = 3 \times 47 \times 70922$, then $10^7 = -2 + Mp$, whose square and cube &c. give

$$10^{14} = 4 + M'p, \quad 10^{21} = -8 + M''p,$$

so that, by now dividing N from right to left into periods a, b, c, d, each of which consists of seven figures,—but the last period, as mentioned already, may contain more or less than seven figures,—then, as before,

$$N = a + 10^7 b + 10^{14} c + 10^{21} d,$$

and so the remainder of $\frac{N}{47}$ must = the remainder of $\frac{a - 2b + 4c - 8d}{47}$, &c.

4505. (Proposed by Sir JAMES COCKLE, F.R.S.)—Find a complete solution of the partial biordinal

$$q^2 r - A^2 t + \frac{A^2}{\eta}\frac{d\eta}{dy} q = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

η being a function of y.

Solution by the PROPOSER.

1. The symbolical decomposition of (1) $\div q$ is

$$\left(q\frac{d}{dx} + A\frac{d}{dy}\right)(p - A\log q + A\log\eta) = 0 \quad \ldots\ldots\ldots\ldots \quad (2),$$

whereof the symbolic factor is deduced from the Mongian $q\,dy - A\,dx$ by changing dy into $\frac{d}{dx}$ and dx into $-\frac{d}{dy}$.

2. The solution of (2) depends on that of the system

$$\frac{dz}{dy}\frac{dU}{dx} + A\frac{dU}{dy} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$\frac{dz}{dx} - A\log\frac{dz}{dy} + A\log\eta = U \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

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3. Assume $z = \theta Y + \Theta$, where θ and Θ are functions of x, and Y is a function of y. Then (3) gives

$$\theta \frac{dY}{dy}\frac{dU}{dx} + A\frac{dU}{dy} = 0, \text{ and } U = \phi\left(Y - A\int\frac{dx}{\theta}\right).$$

4. Take $$U = a\left(Y - A\int\frac{dx}{\theta}\right) + g.$$

Then an arbitrary constant added after the integration would be redundant, and (4) gives

$$\left(\frac{d\theta}{dx} - a\right)Y - A\log\frac{dY}{dy} + A\log\eta = A\log\theta - Aa\int\frac{dx}{\theta} - \frac{d\Theta}{dx} + g \ldots (5).$$

Let $\frac{d\theta}{dx} - a = b$, then

$$\theta = (a+b)x + c \text{ and } \int\frac{dx}{\theta} = \frac{\log\theta}{a+b},$$

and (5) will be satisfied if

$$bY - A\log\frac{dY}{dy} + A\log\eta = f \ldots\ldots\ldots\ldots (6),$$

$$\frac{Ab\log\theta}{a+b} + g - \frac{d\Theta}{dx} = f \ldots\ldots\ldots\ldots (7).$$

5. Now (6) gives $$Y = -\frac{A}{b}\log\left(-\frac{b}{A}e^{-\frac{f}{A}}\int\eta dy\right);$$

also (7) gives $$\Theta = \frac{Ab}{(a+b)^2}(\theta\log\theta - \theta) - (f-g)x + h.$$

Put $a+b=b\beta$, $c=b\gamma$, and $\theta = b\vartheta = b(\beta x + \gamma)$. Then

$$\theta Y = -A\vartheta\left\{\log\int\eta dy - \log(-A) + \log b - \frac{f}{A}\right\},$$

and $$\Theta = \frac{A}{\beta^2}(\vartheta\log\theta - \vartheta) - \frac{f-g}{\beta}(\vartheta - \gamma) + h.$$

Consequently $$z = -A\vartheta\left\{\log\int\eta dy - \frac{\log\vartheta}{\beta^2} + i\right\} + j \ldots\ldots\ldots\ldots (8),$$

where $$i = -\log(-A) + \left(1 - \frac{1}{\beta^2}\right)\log b + \frac{f}{A}\left(\frac{1}{\beta} - 1\right) + \frac{1}{\beta^2} - \frac{A\beta}{g}$$

and $$j = \frac{(f-g)\gamma}{\beta} + h.$$

Hence z contains five arbitrary constants, viz., β, γ, i, j, and that given by $\int\eta\,dy$. Therefore the solution is complete.

6. We also have $$U = aY - \frac{Aa\log\theta}{a+b} + g$$

$$= A(1-\beta)\left\{\log\int\eta dy - \log(-A) + \log b - \frac{f}{A}\right\} + \frac{A(1-\beta)\log\theta}{\beta} + g$$

$$= A(1-\beta)\left\{\log\int\eta dy + \frac{\log\vartheta}{\beta} + \left(1 + \frac{1}{\beta}\right)\log b - \log(-A) - \frac{f}{A}\right\} + g.$$

7. Now $$p = -A\beta\left\{\log\int\eta dy - \frac{\log\vartheta}{\beta^2} + i\right\} + \frac{A}{\beta} \text{ and } q = -A\vartheta\frac{\eta}{\int\eta dy},$$

as we see by (8).

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Also $$A\log q = A\left\{\log(-A)+\log\mathfrak{Z}+\log\eta-\log\int\eta\,dy\right\}.$$

Therefore
$$p-A\log q$$
$$= A(1-\beta)\left\{\log\int\eta\,dy+\frac{\log\mathfrak{Z}}{\beta}\right\}-A\log\eta-A\log(-A)+\frac{A}{\beta}(1-i\beta^2).$$

Hence, if (4) is satisfied,

$$\frac{A}{\beta}(1-\beta^2)\log b+A\beta\log(-A)-(1-\beta)f+g-\frac{A}{\beta}+Ai\beta=0.$$

But the value of i, given in Art. 5, satisfies this relation. I shall give another verification.

7. Let $\eta=1$, $a=0$, and $g=0$. Then $\beta=1$, $U=0$, $\mathfrak{Z}=x+\gamma$, and we may put

$$z=-A(x+\gamma)\left\{\log\frac{y+k}{x+\gamma}-\log(-A)+1\right\}=-A(x+\gamma)\left\{\log\frac{e(y+k)}{-A(x+\gamma)}\right\}.$$

Now if we replace $-A$ by A, the result ought to be capable of coinciding with the z of my solution of Question 4265, for the equation there dealt with is $p+A\log q=0$. But if in the last preceding value of z we write A for $-A$, $-c$ for γ, and Ab for ek, it becomes $z=A(x-c)\log\frac{ey+Ab}{A(x-c)}$ and does so coincide.

8. If we take the general value of z, then $q\,dy-A\,dx=0$ gives

$$\frac{\eta\,dy}{\int\eta\,dy}+\frac{dx}{\mathfrak{Z}}=0 \text{ and } \log\int\eta\,dy+\frac{1}{\beta}\log\mathfrak{Z}=\text{const.}$$

Call the sinister of this equation u. Then U is a function of u and of constants.

4532. (Proposed by Professor TOWNSEND, F.R.S.)—Four material particles a, b, c, d, connected with a common point O by four inextensible cords OA, OB, OC, OD, repel each other with forces varying directly as their masses and mutual distances conjointly; shew that, in their configuration of relative equilibrium,

$$BCDO : CDAO : DABO : ABCO = a : b : c : d,$$

each tetrad of letters representing the volume of the tetrahedron of which its constituents are the vertices.

I. *Solution by* Professor WOLSTENHOLME, M.A.

The three repulsive forces from b, c, d on a will have a resultant acting from the centre of inertia of b, c, d, which must therefore lie in the straight line OA; hence the centre of inertia of a, b, c, d must lie in the line OA, and similarly in OB, &c., or it must be the point O.

Hence the distance of this centre from the plane BCD : distance of A from the plane BCD $= a : a+b+c+d$; therefore

$$\frac{\text{vol. OBCD}}{a}=\frac{\text{vol. ABCD}}{a+b+c+d}, \text{ and similarly } =\frac{\text{OCDA}}{b}=\frac{\text{ODAB}}{c}=\frac{\text{OABC}}{d}.$$

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II. *Solution by* J. J. WALKER, M.A.; J. M. JOHNSON, B.A.; *and others.*

Let β, γ, δ be the perpendiculars from B, C, D on OA; then for the equilibrium of the particle a there must be equilibrium among forces acting on A parallel and proportional to $b\beta$, $c\gamma$, $d\delta$. Hence, by Lami's principle, we have $b\beta : c\gamma = \sin\gamma\delta : \sin\beta\delta$,

or $b : c = \gamma . \sin\gamma\delta : \beta . \sin\beta\delta = \text{ACO} . \sin\gamma\delta : \text{ABO} . \sin\beta\delta$.

But these sines are proportional to the perpendiculars from D on the planes of the triangles ACO, ABO respectively; hence

$b : c = \text{ACDO} : \text{ABDO}$; therefore &c. &c.

[Professor TOWNSEND remarks that, the point O being evidently, under the circumstances supposed, the centre of gravity of the four masses a, b, c, d, therefore &c.]

III. *Solution by* JOHN C. MALET, M.A.

Since the point O is at rest, the resultant of the tensions along OC and OD lies in the plane ABO. Hence, if we resolve the forces acting on the system ABO perpendicular to the plane ABO, calling the perpendiculars on this plane from C and D, p_3 and p_4, we have for equilibrium

$$(ac + bc)\, p_3 = (ad + bd)\, p_4, \quad \text{or} \quad cp_3 = dp_4;$$

therefore $c : d = \text{ABDO} : \text{ABCO}$,

with similar relations for any two of the points. Hence, finally,

$$a : b : c : d = \text{BCDO} : \text{ACDO} : \text{ABDO} : \text{ABCO}.$$

A NEW METHOD OF TREATING BIQUADRATIC EXPRESSIONS WITH A VIEW TO RENDERING THEM SQUARES. *By* S. BILLS.

Consider the expression $x^4 + 4x^3 + 8x^2 + 7x + 6 = \square$ .....(A), in which the first term is a square.

Now (A) will be a square, viz. $= (\pm 2)^2$, when $x = -1$. Assume $(\text{A}) = (x^2 + 2x + 3)^2$; this root being so assumed as to be $= +2$ when $x = -1$, and also to make the first two terms of (A) definite. Squaring the above, we shall have $x^4 + 4x^3 + 8x^2 + 7x + 6 = x^4 + 4x^3 + 10x^2 + 12x + 9$; whence we obtain the quadratic $2x^2 + 5x + 3 = 0$. Now we know that one root of this quadratic is $x = -1$; therefore the other will be $x = -\frac{3}{2}$, which will give another solution of (A); that is, $x = -\frac{3}{2}$ will render (A) a square. Again, assume $(\text{A}) = (x^2 + 2x - 1)^2$; this root being so taken as to $= -2$ when $x = -1$, and also to make the first two terms of (A) vanish. Treating this as before, we get the quadratic $6x^2 + 11x + 5 = 0$. We know that $x = -1$ is one root, therefore $x = -\frac{5}{6}$ will be the other, which will give another solution of (A). It will thus be seen that, from one known solution of (A), we are enabled to find two others without rising to large numbers. Of course the same method is applicable when the last term is a square.

Even when the known value of x is obtained by Euler's method, that is, by making the first three terms vanish, we can by this means obtain one other solution. For instance, $x = 2$ is Euler's solution, which makes

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(A) $= 100 = (\pm 10)^2$. Assume (A) $= (x^2+2x-18)^2 = (-10)^2$ when $x=2$. From this we get the quadratic $40x^2+79x-318=0$, whence $x=2$, or $x=-\frac{159}{40}$. When both the first and last terms of the given expression are squares, no less than eight other solutions may be deduced from one known solution by this method. Take, for example, the expression

(B) ..... $x^4+4x^3+8x^2+7x+16 = \square = (\pm 6)^2$, when $x=1$.

Take (B) $= (x^2+2x+3)^2 = (+6)^2$, when $x=1$;
then we get $2x^2+5x-7=0$; whence $x=1$, $x=-\frac{7}{2}$.
Again, take (B) $= (x^2+2x-9)^2 = (-6)^2$, when $x=1$;
whence $22x^2+43x-65=0$; $x=1$, $x=-\frac{65}{22}$.
Take (B) $= (4+\frac{7}{8}x+\frac{2}{8}x^2)^2 = (+6)^2$, when $x=1$;
then we obtain $17x^2-130x+113=0$; $x=1$, $x=-\frac{113}{17}$.
Again take (B) $= (4+\frac{7}{8}x-\frac{6}{8}x^2)^2 = (-6)^2$, when $x=1$;
developing this and reducing as before, we find

$$x=1 \text{ and } =-\frac{6031}{7505}.$$

Again, taking (B), successively,

$= (x^2+x+4)^2$, $= (x^2-3x-4)^2$, $= (x^2+9x-4)^2$, $= (x^2-11x+4)^2$,

we find $x=1$; also $x=-\frac{1}{2}$, $x=-\frac{17}{10}$, $x=-\frac{79}{14}$, $x=\frac{95}{26}$.

Any other biquadratic expression in which either the first or last term, or both, are squares, may be treated in a similar manner, when we know one satisfactory value of x.

By this method we may frequently obtain solutions to problems in much *smaller numbers* than by the ordinary processes.

4445. (Proposed by E. B. Elliott, B.A.)—In the equation $ax^2+2hxy+by^2=1$, a, h and b may have any real values, all being equally likely. Show that the chances of the conic represented in rectangular coordinates being an ellipse, an imaginary conic, or an hyperbola, are respectively $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$; and in the case of an hyperbola distinguish between the cases in which the curve meets in real points, both coordinate axes (chance $\frac{5}{36}$), neither axis ($\frac{5}{36}$), or one and only one ($\frac{1}{4}$).

I. *Solution by* A. M. Nash, B.A.

If a and b are of opposite signs, the curve is an hyperbola; and the chance of this is $\frac{1}{2}$. If a and b are of the same sign, the curve is an hyperbola if $h^2>ab$. If $h^2<ab$, the curve is an ellipse if a and b are both positive (chance $=\frac{1}{4}$); and an imaginary conic if they are both negative (chance $=\frac{1}{4}$). Suppose a, h, b to be all less than n, then the chance that $h^2<ab$ is

$$\int_0^n\int_0^n\int_0^{\sqrt{ab}} da\,db\,dh \div \int_0^n\int_0^n\int_0^n da\,db\,dh = \tfrac{4}{9};$$

therefore the chance that the curve is an ellipse is $\frac{4}{9}\times\frac{1}{4}=\frac{1}{9}$. The chance that the curve should be an imaginary conic is also $\frac{1}{9}$. Hence the chance

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that it should be an hyperbola is $\frac{7}{9}$. If a and b are of opposite signs, the hyperbola only cuts one axis (chance $= \frac{1}{2}$). If a and b are both positive, it meets both axes [chance $= (\frac{7}{9} - \frac{1}{2}) \times \frac{1}{2} = \frac{5}{36}$]. If a and b be both negative, it meets neither axis, and the chance is $\frac{5}{36}$.

II. *Solution by the* PROPOSER.

The sign of h has nothing to do with the species of conic. In order that the conic may be an ellipse, a and b must be both positive and $h^2 < ab$; hence the chance of an ellipse is

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \int_0^c \int_0^c \int_0^{\sqrt{ab}} dh\, da\, db \div \int_0^c \int_0^c \int_0^c dh\, da\, db \quad (\text{where } c = \infty)$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4c^3}{9} \div c^3 = \frac{1}{9} \quad \ldots\ldots (1).$$

For an imaginary conic, a and b must be both negative, while $h^2 < ab$; therefore the chance of an imaginary conic is

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \int_{-c}^0 \int_{-c}^0 \int_0^{\sqrt{ab}} dh\, da\, db \div c^3 = \frac{1}{9} \quad \ldots\ldots (2).$$

For an hyperbola meeting both axes, a and b must be positive and $h^2 > ab$; therefore the chance of such an hyperbola is

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \int_0^c \int_0^c \int_{\sqrt{ab}}^c dh\, da\, db \div c^3 = \frac{1}{4}\left(1 - \frac{4}{9}\right) = \frac{5}{36} \quad \ldots\ldots (3).$$

Similarly the chance of an hyperbola meeting neither axis, that is, the chance that both a and b be negative and $h^2 > ab$ is

$$\frac{1}{4}\left(1 - \frac{4}{9}\right) = \frac{5}{36} \quad \ldots\ldots (4).$$

The conic will be an hyperbola meeting one axis, and only one, if a and b are of different signs, the chance of which is $\frac{1}{2}$ ..... (5).

4463. (Proposed by the EDITOR.)—Let a, b be two conjugate semidiameters of an ellipse; and (x', y') the coordinates, in reference thereto, of a variable point in the curve: show that the envelope of a series of ellipses whose semidiameters are coincident in direction with a, b, and in magnitude are mean proportionals between a, x' and b, y', is given by the projective or tangential equation

$$\left(\frac{x}{a}\right)^{\frac{4}{3}} + \left(\frac{y}{b}\right)^{\frac{4}{3}} = 1, \quad \text{or} \quad a^4\xi^4 + b^4\upsilon^4 = 1.$$

I. *Solution by the* Rev. J. R. WILSON, M.A.; R. TUCKER, M.A.; *and others.*

Let (x, y) be the coordinates of a point on one of the ellipses; then, by

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the question, we have $\frac{x^2}{ax'} + \frac{y^2}{by'} = 1$, $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$ ..... (1, 2).

Differentiating (1) and (2) in the usual way, multiplying the second equation so obtained by an indeterminate multiple λ, and adding, we have

$$\left(\lambda\frac{x}{a^2} - \frac{x^2}{ax'^2}\right)Dx' + \left(\lambda\frac{y'}{b^2} - \frac{y^2}{by'^2}\right)Dy' = 0;$$

therefore $$\frac{\lambda x'}{a^2} - \frac{x^2}{ax'^2} = 0, \quad \text{and} \quad \frac{\lambda y'}{b^2} - \frac{y^2}{by'^2} = 0.$$

Multiplying the first of these equations by x' and the second by y', and adding, we have, by (1) and (2), $\lambda - 1 = 0$ or $\lambda = 1$; hence $x'^3 = ax^2$, $y'^3 = by^2$; and then, by substituting these values of x', y' in (2), there results for the required envelope the equation given in the question.

II. *Solution by the* Rev. Dr. Booth, F.R.S.

The squares of the semi-diameters of the derived ellipse are ax' and by'; hence the tangential equation of this ellipse is $ax'\xi^2 + by'v^2 = 1$. And the equation of the given ellipse is $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$.

Eliminating y' between these equations, we shall have

$$\frac{x'^2}{a^2} + \frac{(1 - a\xi^2x')^2}{b^4v^4} - 1 \equiv V = 0.$$

Hence $\frac{dV}{dx'} = 0$ gives $$\frac{x'}{a} = \frac{a^2\xi^2}{a^4\xi^4 + b^4v^4};$$

therefore $$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 = \frac{a^4\xi^4 + b^4v^4}{(a^4\xi^4 + b^4v^4)^2}, \text{ or } a^4\xi^4 + b^4v^4 = 1.$$

[This solution embodies an important principle. It shows how the tangential method may be extended to those cares in which the envelope is generated by the successive intersections of curves whose parameters vary according to a given law. Thus, in this case, we require three equations to eliminate x' and y', namely the equation of the given ellipse $x\xi + yv = 1$ and $\frac{dV}{dx'} = 0$.

The formulæ of transition by which we may pass from the projective to the tangential equation of a curve, or *vice versâ*, are given in Dr. Booth's *New Geometrical Methods*, p. 14; that to say, if F (x, y) be the projective equation of a curve, we have

$$\xi = \frac{dF}{dx} \div \left(\frac{dF}{dx}x + \frac{dF}{dy}y\right), \quad v = \frac{dF}{dy} \div \left(\frac{dF}{dx}x + \frac{dF}{dy}y\right).$$

Applying this formula to the projective equation obtained in the first solution, we obtain

$$\frac{dF}{dx} = \frac{4}{3a}\left(\frac{x}{a}\right)^{\frac{1}{3}}, \quad \frac{dF}{dy} = \frac{4}{3b}\left(\frac{x}{b}\right)^{\frac{1}{3}}, \quad \frac{dF}{dx}x + \frac{dF}{dy}y = \frac{4}{3};$$

hence $$a\xi = \left(\frac{x}{a}\right)^{\frac{1}{3}}, \quad \text{and} \quad bv = \left(\frac{y}{b}\right)^{\frac{1}{3}},$$

therefore $$(a\xi)^4 + (bv)^4 = 1,$$

which gives the tangential equation obtained in Dr. Booth's solution.]

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4507. (Proposed by the Rev. Dr. Booth, F.R.S.)—Eliminate x, y, z from $D\xi = Ax + B_1z + B_2y + C$, $Dv = A_1y + B_2x + Bz + C_1$,
$D\zeta = A_2z + By + B_1x + C_2$, and $x\xi + yv + z\xi = 1$,
where $D = 1 - Cx - C_1y - C_2z$;
or, in other words, given the projective equation of a surface of the second order, find its tangential equation referred to the same axes.

Solution by the Editor.

1. Let
$$Ax^2 + A_1y^2 + A_2z^2 + 2Byz + 2B_1zx + 2B_2xy + 2Cx + 2C_1y + 2C_2z = 1 \quad \ldots (1)$$
be the projective equation of a surface of the second order referred to three rectangular axes in space; then the tangential coordinates (ξ, v, ζ) of the plane touching the surface in the point (x, y, z) are found by using the *Equations of Transition* given in Art. 80 of Dr. Booth's *New Geometrical Methods*, which in this case take the forms given in the question.

As my elimination of x, y, z from these equations is very tedious, I shall here put down the result only, which is as follows:—

$$\begin{aligned}
&AB^2 + A_1B_1^2 + A_2B_2^2 - AA_1A_2 - 2BB_1B_2 \\
= \;& \{(B^2 - A_1A_2) + 2BC_1C_2 - A_2C_1^2 - A_1C_2^2\}\,\xi^2 \\
+ \;& \{(B_1^2 - AA_2) + 2B_1CC_2 - AC_2^2 - A_2C^2\}\,v^2 \\
+ \;& \{(B_2^2 - AA_1) + 2B_2CC_1 + A_1C^2 - AC_1^2\}\,\zeta^2 \\
+ \;& 2\{(AB - B_1B_2) + (BC - B_2C_2)C + (AC_2 - CB_1)C_1\}\,v\zeta \\
+ \;& 2\{(A_1B_1 - BB_2) + (B_1C_1 - BC)C_1 + (A_1C - C_1B_2)C_2\}\,\zeta\xi \\
+ \;& 2\{(A_2B_2 - B_1B) + (B_2C_2 - B_1C_1)C_2 + (A_2C_1 - C_2B)C\}\,\xi v \\
+ \;& 2\{(A_1A_2 - B^2)C + (BB_1 - A_2B_2)C_1 + (B_2B - A_1B_1)C_2\}\,\xi \\
+ \;& 2\{(AA_2 - B_1^2)C_1 + (B_1B_2 - AB)C_2 + (BB_1 - A_2B_2)C\}\,v \\
+ \;& 2\{(AA_1 - B_2^2)C_2 + (BB_2 - A_1B_1)C + (B_1B_2 - AB)C_1\}\,\zeta \quad \ldots\ldots(2).
\end{aligned}$$

This is the tangential equation of the surface (1) referred to the same axes of coordinates.

2. When the surface is referred to axes whose origin is at the centre, we have $C = C_1 = C_2 = 0$, and the tangential equation becomes
$$\begin{aligned}
&AB^2 + A_1B_1^2 + A_2B_2^2 - AA_1A_2 - 2BB_1B_2 \\
= \;& (B^2 - A_1A_2)\xi^2 + (B_1^2 - AA_2)v^2 + (B_2^2 - AA_1)\zeta^2 \\
+ \;& 2(AB - B_1B_2)v\zeta + 2(A_1B_1 - BB_2)\zeta\xi + 2(A_2B_2 - B_1B)\xi v \quad \ldots(3).
\end{aligned}$$

3. Let X, Y, Z be the projective coordinates of the centre of the surface, then (Booth's *New Geometrical Methods*, Art. 75) we have
$$X = \frac{(A_1A_2 - B^2)C + (BB_1 - A_2B_2)C_1 + (B_2B - A_1B_1)C_2}{AB^2 + A_1B_1^2 + A_2B_2^2 - AA_1A_2 - 2BB_1B_2},$$
with like expressions for Y and Z.

4. When the surface is a paraboloid, $X = \infty$, and we have
$$AB^2 + A_1B_1^2 + A_2B_2^2 - AA_1A_2 - 2BB_1B_2 = 0 \quad \ldots\ldots\ldots\ldots (4),$$
a well-known relation.

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4420. (Proposed by J. L. McKenzie.)—Prove that in a system of confocal ellipses, the envelope of a normal that makes with the major axis an angle whose sine is bc^{-1}, is a four-cusped hypocycloid, with two opposite cusps at the foci of the system.

I. *Solution by* R. Tucker, M.A.

The equation to a normal is

$$ax \sin\phi - by\cos\phi = c^2 \sin\phi\cos\phi \quad \ldots\ldots\ldots\ldots (1).$$

From the given data we have

$$\tan\phi = \frac{b^2}{a(c^2-b^2)^{\frac{1}{2}}}, \quad \sin\phi = \frac{b^2}{c^2}, \quad \cos\phi = \frac{a(c^2-b^2)^{\frac{1}{2}}}{c^2}.$$

Substituting in (1), we have to find the envelope of

$$bx = (c^2-b^2)^{\frac{1}{2}}(y+b) \quad \ldots\ldots\ldots\ldots (2).$$

Differentiating, we get $(c^2-b^2)^{\frac{1}{2}}x + by = c^2-2b^2$ ..... (3).

Eliminating x from (2) and (3), we have $c^2y = -b^3$ ..... (4);

hence, by (2), $bc^2x = (c^2-b^2)^{\frac{1}{2}}(bc^2-b^3)$, *i. e.*, $c^2x = (c^2-b^2)^{\frac{1}{2}}$ ..... (5),

whence, from (4) and (5), we have $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$, which is a hypocycloid fulfilling the stated conditions.

II. *Solution by the* Editor.

The tangential equation of the evolute of an ellipse is (see Dr. Booth's *New Geometrical Methods*, Art. 172)

$$a^2v^2 + b^2\xi^2 = (a^2-b^2)\xi^2v^2, \text{ or } b^2(\xi^2+v^2) + c^2v^2 = c^4\xi^2v^2 \quad \ldots\ldots (1).$$

But, by the condition assumed, we have

$$\sin^2\phi = \frac{b^2}{c^2} = \frac{\xi^2}{\xi^2+v^2}, \text{ or } b^2(\xi^2+v^2) = c^2\xi^2 \quad \ldots\ldots (2);$$

hence, eliminating b between (1) and (2), we get

$$\xi^2 + v^2 = c^2\xi^2v^2 \quad \ldots\ldots (3),$$

which is the tangential equation of the quadrantal hypocycloid (Booth's *New Geometrical Methods*, Art. 134).

III. *Solution by the* Proposer.

The tangential coordinates of a normal to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ must satisfy the condition $a^2v^2 + b^2\xi^2 = c^4\xi^2v^2$, (the equation of the evolute.) The condition that the normal should make the given angle with the major axis, gives $-\xi = v\tan\theta = vb(c^2-b^2)^{-\frac{1}{2}}$, or $b^2\xi^2 = c^2\xi^2 - b^2v^2$. Hence $a^2v^2 + b^2\xi^2 = c^2(\xi^2+v^2)$; and combining this with the equation of the evolute, we get for the required envelope, $\xi^2 + v^2 = c^2\xi^2v^2$, a hypocycloid of the kind stated in the question. (See Booth's *New Geometrical Methods*, p. 134.)

4405. (Proposed by T. Cotterill, M.A.)—In a spherical triangle, if

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h be the perpendicular from the angle C on the side c, which is bisected internally in F and externally in F', prove that
$\sin^2 \frac{1}{2}c \cos^2 h + \cos a \cos b = \cos^2 \mathrm{FC}$, and $\cos^2 \frac{1}{2}c \cos^2 h - \cos a \cos b = \cos^2 \mathrm{F'C}$.
Hence, find the numerical limits of the expressions $\sin^2 \frac{1}{2}c + \cos a \cos b$, and $\cos^2 \frac{1}{2}c - \cos a \cos b$.

Solution by the PROPOSER.

The forms $\cos a + \cos b = 2 \cos \frac{1}{2}c \cos \mathrm{FC}$, $\cos a - \cos b = 2 \sin \frac{1}{2}c \cos \mathrm{F'C}$, and $\sin^2 c \sin^2 h = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c$, are well known. Hence we have

$$\begin{aligned}
\sin^2 c \sin^2 h &= \sin^2 c - \cos^2 a - \cos^2 b + 2 \cos a \cos b \cos c \\
&= 4 \cos^2 \tfrac{1}{2}c (\sin^2 \tfrac{1}{2}c + \cos a \cos b) - (\cos a + \cos b)^2 \\
&= 4 \sin^2 \tfrac{1}{2}c (\cos^2 \tfrac{1}{2}c - \cos a \cos b) - \cos a - \cos b)^2 \\
&= 4 \cos^2 \tfrac{1}{2}c (\sin^2 \tfrac{1}{2}c + \cos a \cos b - \cos^2 \mathrm{FC}) \\
&= 4 \sin^2 \tfrac{1}{2}c (\cos^2 \tfrac{1}{2}c - \cos a \cos b - \cos^2 \mathrm{F'C}).
\end{aligned}$$

The equations in the question at once follow. Also

$$\cos^2 \mathrm{FC} + \cos \mathrm{F'C} = \cos^2 h;$$

and if p and p' are the perpendiculars from A or B on FC and F'C, we have

$$\begin{aligned}
\cos^2 \tfrac{1}{2}c - \cos a \cos b &= \sin^2 \tfrac{1}{2}c + \sin a \sin b \cos \mathrm{C} \\
&= \cos^2 \tfrac{1}{2}c \sin^2 h + \cos^2 \mathrm{F'C} = \sin^2 \mathrm{FC} - \sin^2 \tfrac{1}{2}c \sin^2 h \\
&= \sin^2 \mathrm{FC} \left\{ 1 - \sin^2 \tfrac{1}{2}c \sin^2 (\mathrm{FC}, c) \right\} = \sin^2 \mathrm{FC} \cos^2 p.
\end{aligned}$$

Similarly $\sin^2 \frac{1}{2}c + \cos a \cos b = \cos^2 \frac{1}{2}c - \sin a \sin b \cos \mathrm{C} = \sin^2 \mathrm{F'C} \cos^2 p$, and the numerical limits of these expressions are 0 and +1. These equations are, however, more easily obtained by applying the form

$$\sin^2 c \cos^2 h = \cos^2 a + \cos^2 b - 2 \cos a \cos b \cos c$$

to a triangle such as CAF.

An important deduction from the last equation follows from throwing it into the form

$$\begin{aligned}
\sin^2 c \cos^2 h &= \cos a (\cos a - \cos b \cos c) + \cos b (\cos b - \cos a \cos c) \\
&= \sin b \sin c \cos a \cos \mathrm{A} + \sin a \sin c \cos b \cos \mathrm{B}.
\end{aligned}$$

Hence, "In a spherical triangle, of the three products $\cos a \cos \mathrm{A}$, $\cos b \cos \mathrm{B}$, $\cos c \cos \mathrm{C}$, two at least are positive."

4480. (Proposed by G. S. CARR.)—A heavy cylinder rotating rapidly on its axis is projected upwards in the direction of its axis, which is inclined to the horizon. Assuming that the resultant pressure of the air upon the cylinder perpendicular to its axis varies as the square of the velocity of the cylinder in that direction, and that the friction against the surface of the cylinder varies as the pressure of the air upon it: show that the distance of the projectile from the vertical plane of projection after the time t will be the same for all initial velocities of projection, and will be

$$c \left\{ \tfrac{1}{2}t^2 + \frac{t}{a} - \frac{1}{a^2} \log \tfrac{1}{2} (e^{2at} + 1) \right\},$$

where c and a are constants.

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The axis of the cylinder is supposed to retain its original direction, and the resistance of the air to the lateral motion itself is left out of the calculation.

Solution by the PROPOSER.

Taking the point of projection for the origin of rectangular axes X, Y, Z, the axis of X in a vertical plane and perpendicular to the line of projection, the velocity parallel to X acquired in the time $t = v$ will be $K\frac{e^{nt}+1}{e^{nt}-1}$ (Tait and Steele, p. 233); writing n for $\frac{2f}{K}$, f being here equal to g resolved parallel to X; and V, the initial velocity parallel to X, being zero.

Now the axis of the cylinder remaining parallel to Y, the compressed air in advance of the cylinder will produce a resultant force of friction perpendicular to XY and proportional to v^2. Thus the equation of motion in the Z direction will be

$$\frac{d^2z}{dt^2} = c\left(\frac{e^{nt}-1}{e^{nt}+1}\right)^2 = c\left\{1 - \frac{4e^{nt}}{(e^{nt}+1)^2}\right\}, \quad \therefore \frac{dz}{dt} = c\left\{t + \frac{4}{n(e^{nt}+1)} - \frac{2}{n}\right\},$$

since $\frac{dz}{dt}$ and t are each zero at starting. Integrating again, observing that

$$\int \frac{dt}{e^{nt}+1} = t - \frac{1}{n}\log(e^{nt}+1),$$

and that $z=0$ when $t=0$, there results the equation in the Question.

The constants c and a depend upon the density of the air, the coefficient of friction, the mass of the projectile, and the angle of projection; but the initial velocity of projection being at right angles to Z, will not enter into the value of v, and therefore will not affect the result.

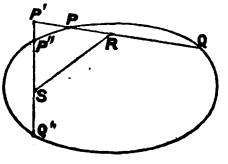
3912. (Proposed by R. W. GENESE, B.A.)—Prove (1) that any chord PQ of a conic is to the parallel chord through the focus S as SP is to ST, T being the point in which the parallel meets the tangent at P; and hence show (2) that the chord of curvature through the focus of a conic is equal to the focal chord parallel to the tangent.

Solution by the PROPOSER.

1. Let any focal chord P″Q″ meet PQ produced in P′. Then

$$P'P \cdot P'Q : P'P'' \cdot P'Q'' = F : P''Q'',$$

where F is the focal chord parallel to PQ. Draw SR parallel to P″P; then P′P : P′P″ = P′R : P′S. Now let P′ move up to coincidence with P, then SR is parallel to the tangent at P, and PR is the ST of the question; therefore



$$ST \cdot PQ : SP \cdot P''Q'' = F : P''Q'' = F \cdot SP : P''Q'' \cdot SP;$$

therefore $ST \cdot PQ = F \cdot SP$, or $PQ : F = SP : ST$.

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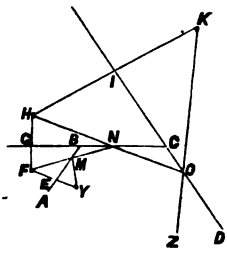
4283. (Proposed by T. T. WILKINSON, F.R.A.S.)—A, B, C, D, E, &c., is an irregular unclosed polygon, and Y, Z are two given points within it. It is required to draw YM, MN, NO, &c., meeting the sides AB, BC, CD, &c., at M, N, O, &c., and making the angle YMA = BMN, the angle MNB = CNO, &c., so that the last line shall pass through the given point Z.

Solution by the PROPOSER.



Draw YF cutting AB at right angles in E and making EF = YE; draw FH cutting CB in G and making GH = FG; draw HK cutting DC in I and making IK = HI; draw KZ cutting CD in O; also draw HO cutting BC in N, and FN cutting AB in M; and, lastly, join M, Y. Then YM, MN, NO, OZ are drawn as required by the question. For it is evident that each of the triangles MFY, NHF, OKH is isosceles, and each of their angles FMY, HNF, KOH is respectively bisected by ME, NG, OI. Hence $\angle AMY = AMF = BMN$; also $\angle BNM = BNH = CNO$; and $\angle CON = COK = DOZ$. Consequently $\angle AMY = BMN$, $\angle BNM = CNO$, and $\angle CON = DOZ$. It is further manifest that this method applies to a polygon of any number of sides.

4554. (Proposed by Professor WOLSTENHOLME, M.A.)—Prove that

$$\int_0^{\frac{1}{2}\pi} \log \sin x \,.\, \log \cos x \, dx = \tfrac{1}{4}\pi (\log 2)^2 - \tfrac{1}{96}\pi^3 \quad \ldots\ldots\ldots\ldots (1);$$

$$\int_0^{\frac{1}{2}\pi} \log (\cos x + \sin x) \log (\cos x - \sin x)\, dx = \tfrac{1}{16}\pi (\log 2)^2 - \tfrac{1}{96}\pi^3 \ldots (2).$$

Solution by the PROPOSER.

1. $$U \equiv \cos 2x + \tfrac{1}{2}\cos 4x + \tfrac{1}{3}\cos 6x + \ldots\ldots \text{ to } \infty$$
$$= -\tfrac{1}{2}\log 2(1 - \cos 2x), \quad \text{if } 2x \text{ lie between 0 and } \pi,$$

$$V \equiv \cos 2x - \tfrac{1}{2}\cos 4x + \tfrac{1}{3}\cos 6x - \ldots\ldots \text{ to } \infty$$
$$= \tfrac{1}{2}\log 2(1 + \cos 2x), \quad \text{if } 2x \text{ lie between 0 and } \pi;$$

then $$\int_0^{\frac{1}{2}\pi} UV\,dx = \tfrac{1}{4}\pi \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots\right) = \tfrac{1}{4}\pi (\tfrac{1}{8}\pi^2 - \tfrac{1}{24}\pi^2) = \tfrac{1}{48}\pi^3,$$

since $$\int_0^{\frac{1}{2}\pi} \cos 2rx \cos 2sx \, dx = 0 \quad \text{when } r, s \text{ are unequal integers,}$$

and $$= \tfrac{1}{4}\pi \text{ when } r, s \text{ are equal integers;}$$

therefore $$\int_0^{\frac{1}{2}\pi} \log 2 \sin x \log 2 \cos x \, dx = -\tfrac{1}{96}\pi^3$$

$$= \int_0^{\frac{1}{2}\pi} (\log 2 + \log \cos x)(\log 2 + \log \sin x)\, dx,$$

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for $\int_0^{\frac{1}{2}\pi} \log 2\sin x \log 2\cos x\, dx = \int_{\frac{1}{2}\pi}^{\pi} \log 2\sin x \log 2\cos x\, dx$

$= \frac{1}{2}\int_0^{\pi} \log(2\sin x)\log(2\cos x)\, dx,$

or $-\frac{1}{96}\pi^3 = \frac{1}{4}\pi(\log 2)^2 + \log 2\int_0^{\frac{1}{2}\pi}(\log\sin x + \log\cos x)\, dx$

$+\int_0^{\frac{1}{2}\pi} \log\sin x \log\cos x\, dx,$

and $\int_0^{\frac{1}{2}\pi}(\log\sin x + \log\cos x)\, dx = \int_0^{\frac{1}{2}\pi} \log(\sin x\cos x)\, dx$

$= \frac{1}{2}\int_0^{\pi} \log\left(\frac{\sin x}{2}\right) dx = \frac{1}{2}(-\frac{1}{2}\pi\log 2 - \frac{1}{2}\pi\log 2) = -\frac{1}{2}\pi\log 2,$

or $-\frac{1}{96}\pi^3 = \frac{1}{4}\pi(\log 2)^2 - \frac{1}{2}\pi(\log 2)^2 + \int_0^{\frac{1}{2}\pi} \log\sin x \log\cos x\, dx;$

whence $\int_0^{\frac{1}{2}\pi} \log\sin x \log\cos x\, dx = \frac{1}{2}\int_0^{\pi} \log\sin x \log\cos x\, dx$

$= \frac{1}{4}\pi[(\log 2)^2 - \frac{1}{24}\pi^2].$

2. Hence $\int_0^{\frac{1}{2}\pi} \log(1-\cos x)\log(1+\cos x)\, dx$

$= 2\int_0^{\frac{1}{2}\pi} \log(2\sin^2 x)\log(2\cos^2 x)\, dx$

$= 2\int_0^{\frac{1}{2}\pi} (\log 2 + 2\log\sin x)(\log 2 + 2\log\cos x)\, dx$

$= 2\left\{\frac{1}{4}\pi(\log 2)^2 + 2\log 2(-\frac{1}{2}\pi\log 2) + 4\cdot\frac{1}{4}\pi[(\log 2)^2 - \frac{1}{24}\pi^2]\right\}$

$= \frac{1}{2}\pi[(\log 2)^2 - \frac{1}{6}\pi^2] = -\frac{1}{2}\pi[\frac{1}{6}\pi^2 - (\log 2)^2]$

$= \int_0^{\pi} \log(1-\sin x)\log(1+\sin x)\, dx,$

or $\int_0^{\pi} \log(1-\cos x)\log(1+\cos x)\, dx = -\pi[\frac{1}{6}\pi^2 - (\log 2)^2]$

$= \int_0^{\pi} \log(1-\sin x)\log(1+\sin x)\, dx$

$= 16\int_0^{\frac{1}{2}\pi} \log(\cos x - \sin x)\log(\cos x + \sin x)\, dx.$

Again, $\int_0^{\frac{1}{2}\pi} \log\sin x \log\cos x\, dx = \int_0^{\frac{1}{2}\pi} \log\sin(\frac{1}{4}\pi - x)\log\cos(\frac{1}{4}\pi - x)\, dx$

$= \int_0^{\frac{1}{2}\pi} [\frac{1}{2}\log 2 + \log(\cos x - \sin x)][\frac{1}{2}\log 2 + \log(\cos x + \sin x)]\, dx$

$= \frac{1}{16}\pi(\log 2)^2 + \frac{1}{2}\log 2\int_0^{\frac{1}{2}\pi} \log(\cos^2 x - \sin^2 x)\, dx$

$+\int_0^{\frac{1}{2}\pi} \log(\cos x - \sin x)\log(\cos x + \sin x)\, dx$

$= \frac{1}{16}\pi(\log 2)^2 + \frac{1}{2}\log 2(+\frac{1}{2}\pi\log 2) + \ldots\ldots;$

therefore $\int_0^{\frac{1}{2}\pi} \log(\cos x - \sin x)\log(\cos x + \sin x)\, dx$

$= -\frac{3}{16}\pi(\log 2)^2 + \frac{1}{4}\pi[(\log 2)^2 - \frac{1}{24}\pi^2] = -\frac{1}{16}\pi[\frac{1}{6}\pi^2 - (\log 2)^2].$

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4492. (Proposed by R. F. Scott, B.A.)—Prove that the lines of curvature of the cone $ax^n + by^n + cz^n = 0$, are its generators and the curves of intersection with spheres drawn round the vertex as centre.

Solution by the Proposer.

The lines of curvature on any cone are its intersections with spheres described about the vertex and its generators. For consider the intersection of the cone with one of these spheres. The two surfaces intersect at right angles, and any line on a sphere possesses the properties of a line of curvature, hence (Salmon, p. 266, foot note) the line is a line of curvature on the cone.

In the special case given in the question, the differential equation admits of a simple solution; viz., the differential equation to the lines of curvature is

$$\begin{vmatrix} dx, & dy, & dz \\ ax^{n-1}, & by^{n-1}, & cz^{n-1} \\ ax^{n-2}dx, & by^{n-2}dy, & cz^{n-2}dz \end{vmatrix} = 0, \text{ (Salmon, Art. 204).}$$

Therefore

$$\begin{vmatrix} dx, & dy, & x\,dx + y\,dy + z\,dz \\ ax^{n-1}, & by^{n-1}, & ax^n + by^n + cz^n \\ ax^{n-2}dx, & by^{n-2}dy, & ax^{n-1}dx + by^{n-1}dy + cz^{n-1}dz \end{vmatrix} = 0.$$

The second and third members of the third column vanish since the line lies on the cone. Hence

$$ab\,(xy)^{n-2}\,(x\,dx + y\,dy + z\,dz)\,(x\,dy - y\,dx) = 0.$$

Thus $x^2 + y^2 + z^2 = A$, $x = By$, are the lines of curvature.

4472. (Proposed by J. J. Walker, M.A.)—Show that the line joining any point outside a conic with its centre and the common chord of the two circles drawn through the point, one passing through the points of contact of tangents from it, the other through the foci, are equally inclined to the lines joining the given point with the foci.

I. *Solution by* R. Tucker, M.A.

Let P be the external point; T, T_1 (ϕ, ψ) the points of contact; F, F_1, O the foci and centre respectively;* then the coordinates of P are

$$a\cos\tfrac{1}{2}(\psi+\phi)\sec\tfrac{1}{2}(\psi-\phi), \quad b\sin\tfrac{1}{2}(\psi+\phi)\sec\tfrac{1}{2}(\psi-\phi),$$

and the equations to the circles TPT_1, FPF_1, are

$$x^2 + y^2 = \frac{\cos\frac{1}{2}(\psi+\phi)}{\cos\frac{1}{2}(\psi-\phi)}\left\{\frac{a^2-b^2}{a}\cos\phi\cos\psi + a\right\}x$$

$$+ \frac{\sin\frac{1}{2}(\psi+\phi)}{\cos\frac{1}{2}(\psi-\phi)}\left\{b - \frac{a^2-b^2}{b}\sin\phi\sin\psi\right\}y - (a^2-b^2)\cos(\psi+\phi) \quad \ldots\ldots (1),$$

$$x^2 + y^2 + \frac{(a^2-b^2)\sin\phi\sin\psi - b^2}{b\sin\frac{1}{2}(\phi+\psi)\cos\frac{1}{2}(\psi-\phi)}\,y = a^2e^2 \quad \ldots\ldots\ldots (2).$$

* See diagram to the next Solution.

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The common chord (PP_1) of (1) and (2) is given by

$$x\left\{\frac{a^2-b^2}{a}\cos\phi\cos\psi+a\right\}+y\cot\tfrac{1}{2}(\psi+\phi)\left\{\frac{a^2-b^2}{b}\sin\phi\sin\psi-b\right\}$$
$$=2(a^2-b^2)\cos\tfrac{1}{2}(\phi+\psi)\cos\tfrac{1}{2}(\psi-\phi) \quad \ldots\ldots\ldots\ldots (3).$$

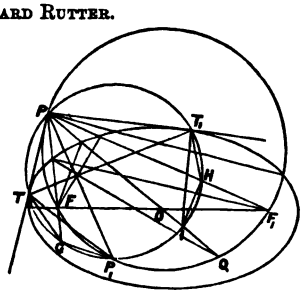
The equations to PT, PT_1, OP are respectively

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1, \quad \frac{x}{a}\cos\psi+\frac{y}{b}\sin\psi=1 \quad \ldots\ldots (4, 5),$$

$$y=-\frac{b}{a}\tan\tfrac{1}{2}(\phi+\psi)\,x \quad \ldots\ldots\ldots\ldots (6).$$

It will be found, in the usual way, that the angle between (3) and (4), that is TPP_1, is equal to the angle between (5) and (6), that is OPT_1; hence, by a property of conics, $\angle OPF_1 = \angle FPP_1$.

II. *Solution by* Edward Rutter.



Let P be the point without the conic; F and F_1 the foci; TT_1 the chord of contact; and TPT_1, FPF_1 the circles of the Question. Then, by construction, we have $\angle T_1FP = \angle TFP$, $\angle T_1F_1P = \angle TF_1P$, $\angle T_1PF_1 = \angle TPF$, $\angle T_1PF = \angle TPF_1$ (see Davies' Hutton, Vol. II., p. 136, Prop. 14). Let PF, PO, PF_1 meet the circle PTT_1 in the points G, I, H. Join the points P_1, G, T, and I, H, T_1; then the quadrilaterals TPP_1G, T_1PIH are in the same circle, and the points P, P_1 common. Hence $\angle TP_1G = \angle TPG = \angle T_1PF_1 = T_1IH$, and $TG = T_1H$; therefore $P_1G = IH$, and $\angle OPF_1 = \angle FPP_1$, which proves the property.

Let PO meet the circle PFF_1 at Q; then $QF = P_1F_1$, $P_1H = IG$, and $QF_1 = P_1F$; therefore GH is parallel to TT_1, and PO bisects both these lines,—a well known property.

4401. (Proposed by A. Martin.)—Find rational triangles whose sides and the lines drawn from the angles to the centre of the inscribed circle shall all be whole numbers.

I. *Solution by* Asher B. Evans, M.A.

Let O be the centre of the inscribed circle; then ABC being a triangle,

$$AO = r\operatorname{cosec}\tfrac{1}{2}A, \quad BO = r\operatorname{cosec}\tfrac{1}{2}B, \quad CO = r\operatorname{cosec}\tfrac{1}{2}C,$$
$$a = r(\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C), \quad b = r(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}C), \quad c = r(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B).$$

Let $\cot\tfrac{1}{2}A = \frac{m}{n}$, $\cot\tfrac{1}{2}B = \frac{p}{q}$;

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then $$\cot \tfrac{1}{2}C = \frac{np+mq}{mp-nq}, \quad \sin \tfrac{1}{2}A = \frac{n}{(m^2+n^2)^{\frac{1}{2}}},$$

$$\sin \tfrac{1}{2}B = \frac{q}{(p^2+q^2)^{\frac{1}{2}}}, \quad \sin \tfrac{1}{2}C = \frac{mp-nq}{\{(m^2+n^2)(p^2+q)\}^{\frac{1}{2}}},$$

where $m^2+n^2 = \square$, $p^2+q^2 = \square$, and r is assumed so as to render AO, BO, CO, a, b, c integral. Let $m=4$, $n=3$, $p=12$, $q=5$, and $r=165$; then AO$=275$, BO$=429$, CO$=325$, $a=676$, $b=500$, $c=616$.

II. *Solution by the* Proposer.

Let a, b, c be the sides of the triangle, r the radius of the inscribed circle, A, B, C the angles of the triangle, and x, y, z the lines drawn from the angles to the centre O of the inscribed circle.

Then AO$=x=r$ cosec $\frac{1}{2}$A, BO$=y=r$ cosec $\frac{1}{2}$B, CO$=z=r$ cosec $\frac{1}{2}$C;

$$BC = a = r(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C), \quad CA = b = r(\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}C),$$

$$AB = c = r(\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B).$$

Take $$\cot \tfrac{1}{2}A = \frac{m}{n}, \text{ and } \cot \tfrac{1}{2}B = \frac{p}{q},$$

then $$\cot^{-1}\left(\frac{m}{n}\right) + \cot^{-1}\left(\frac{p}{q}\right) + \tfrac{1}{2}C = \tfrac{1}{2}\pi,$$

and $$\cot \tfrac{1}{2}C = \frac{(m+n)p+(m-n)q}{(m-n)p-(m+n)q} = \frac{s}{t};$$

$$\cot \tfrac{1}{2}A = \frac{m^2-n^2}{2mn}, \quad \cot \tfrac{1}{2}B = \frac{p^2-q^2}{2pq}, \quad \cot \tfrac{1}{2}C = \frac{s^2-t^2}{2st};$$

$$\text{cosec } \tfrac{1}{2}A = \frac{m^2+n^2}{2mn}, \quad \text{cosec } \tfrac{1}{2}B = \frac{p^2+q^2}{2pq}, \quad \text{cosec } \tfrac{1}{2}C = \frac{s^2+t^2}{2st}.$$

Take $m=3$, $n=1$; $p=4$, $q=1$; then we have

$$s = 9, \quad t = 2, \quad x = \frac{5r}{3}, \quad y = \frac{17r}{8}, \quad z = \frac{85r}{36};$$

$$a = \frac{289r}{72}, \quad b = \frac{125r}{36}, \quad c = \frac{77r}{24}.$$

Now take $r = 72$; and we have

$$x = 120, \quad y = 153, \quad z = 170; \quad a = 289, \quad b = 250, \quad c = 231.$$

4501. (Proposed by Christine Ladd.)—The distance from A to B is a miles. A man at A travels one-mth of the distance to B the first day; the next day, one-nth of the distance back to A; the third day, one-mth of his distance to B; the fourth day, one-nth of the distance back to A, and so on. How far will he be from A at the end of r days (1) when r is even, (2) when r is odd?

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I. *Solution by* A. B. Evans, M.A.; S. Corner; *and others.*

Let $a_1, a_2, a_3 \ldots a_r$ represent the man's distance from A at the end of the first, second, third ... rth days respectively; then his distance from B at the end of the first, second, third ... rth days is $(a-a_1)$, $(a-a_2)$, $(a-a_3)$ ... $(a-a_r)$; hence, putting $1-\frac{1}{n}=p$ and $1-\frac{1}{m}=q$, we have

$$a_1 = \frac{a}{m},$$

$$a_2 = a_1\left(1-\frac{1}{n}\right) = a_1 p = \frac{a}{m}p,$$

$$a_3 = a_2 + \frac{a-a_2}{m} = \frac{a}{m} + a_2\left(1-\frac{1}{m}\right) = \frac{a}{m} + a_2 q = \frac{a}{m}(1+pq),$$

$$a_4 = a_3\left(1-\frac{1}{n}\right) = a_3 p = \frac{a}{m}(1+pq)\,p,$$

$$a_5 = a_4 + \frac{a-a_4}{m} = \frac{a}{m} + a_4\left(1-\frac{1}{m}\right) = \frac{a}{m} + a_4 q = \frac{a}{m}(1+pq+p^2q^2),$$

$$a_6 = a_5\left(1-\frac{1}{n}\right) = \frac{a}{m}(1+pq+p^2q^2)\,p,$$

$$a_7 = a_6 + \frac{a-a_6}{m} = \frac{a}{m} + a_6\left(1-\frac{1}{m}\right) = \frac{a}{m} + a_6 p = \frac{a}{m}(1+pq+p^2q^2+p^3q^3).$$

It is evident from the foregoing that, when r is even,

$$a_r = \frac{a}{m}(1+pq+p^2q^2+p^3q^3+\ldots+p^{\frac{1}{2}(r-2)}q^{\frac{1}{2}(r-2)})\,p = \frac{a}{m}\left(\frac{1-p^{\frac{1}{2}r}q^{\frac{1}{2}r}}{1-pq}\right)p;$$

and when r is odd,

$$a_r = \frac{a}{m}(1+pq+p^2q^2+p^3q^3+\ldots+p^{\frac{1}{2}(r-1)}q^{\frac{1}{2}(r-1)}) = \frac{a}{m}\left(\frac{1-p^{\frac{1}{2}(r+1)}q^{\frac{1}{2}(r+1)}}{1-pq}\right).$$

II. *Solution by the* Proposer.

1. Put $p=1-\frac{1}{m}$, $q=1-\frac{1}{n}$, and $2s=r$ when r is even; and let $x_s=$ his distance from A at the end of r days. Then we have

$$x_{s+1} = pqx_s + (1-p)\,a, \quad \text{or} \quad x_{s+1} - pqx_s = (1-p)\,a \ldots\ldots\ldots\ldots (1);$$

therefore, by integration, $x_s = Cp^sq^s + \frac{(a-p)\,a}{1-pq}$.

When $s=0$, we have $x_s=0$, and $C=-\frac{(1-p)\,a}{1-pq}$;

therefore $$x_s = \frac{(1-p)(1-p^sq^s)\,a}{1-pq}.$$

2. When r is odd, put $2t-1=r$; then $x_t=$ the man's distance from A at the end of r days, and

$$x_{t+1} = pqx_t + q(a-p)\,a, \quad \text{or} \quad x_{t+1} - pqx_t = q(1-p)\,a \ldots\ldots\ldots\ldots (2).$$

Integrating (2), and correcting, $x_t = \frac{q(1-p)(1-p^tq^t)\,a}{1-pq}$.

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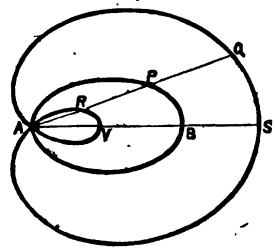
3698. (Proposed by W. Siverly.)—One end of a rod, whose length is equal to the major axis of an ellipse, is inserted through an opening at the extremity of the major axis, and made to pass around the curve; find the curve traced out by the other end of the rod.

Solution by the Editor.

In a more general form, this problem is merely a transference to the ellipse of the well known mode of circular generation of the Limaçon of Pascal.

Thus, on every radius-vector AP, drawn from the vertex A of an ellipse APB, let two points Q, R be taken such that PQ = PR = c; then, when $c < 2a$, the locus of Q and R will be a continuous curve like the one drawn in the annexed diagram, having a loop ARV; where we have AB = $2a$, and BV = BS = c.



The polar equation of the curve is at once written down from the condition $r = \mathrm{AP} \pm c$; that is, putting $\angle \mathrm{PAB} = \theta$, we have

$$r = \frac{2a(1-e^2)\cos\theta}{1-e^2\cos^2\theta} \pm c = \frac{2ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta} \pm c \quad \ldots\ldots\ldots\ldots (1).$$

The corresponding equation in rectangular coordinates (x, y) is readily found to be $(x^2+y^2)(a^2y^2+b^2x^2-2ab^2x)^2 = c^2(a^2y^2+b^2x^2)^2$ ..... (2).

When $c = 2a$, as in the particular case proposed, the loop ARV vanishes, and the curve has a cusp at A; and when $c > 2a$, A is a conjugate point.

When the elliptic generator becomes a circle, or $a = b$, equations (1) and (2) take the respective forms

$$r = 2a\cos\theta \pm c, \text{ and } (x^2+y^2-2ax)^2 = c^2(x^2+y^2) \ldots\ldots (3, 4),$$

and the curve becomes the Limaçon of Pascal.

If, moreover, $c = 2a$, equation (3) takes the simple form

$$r^{\frac{1}{2}} = 2a^{\frac{1}{2}}\cos\tfrac{1}{2}\theta \quad \ldots\ldots\ldots\ldots (5),$$

and the curve becomes the *Cardioid*.

Another solution of this Question is given on pp. 67, 68 of this volume; where, however, the equation of the curve is not in its simplest form, but may readily be put into the form (2) above.

4451. (Proposed by Dr. Hart.)—Find the equation to the curve that will cut at an angle of 45° any number of circles having their centres on a given straight line, and their circumferences passing through a given point in that line.

I. *Solution by* E. B. Elliott, B.A.

Taking the given point and line for origin and axis of x, the equation of the system of circles is $x^2+y^2 = 2ax$, a being arbitrary.

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Thus, $$\frac{dy}{dx} = \frac{a-x}{y} - \frac{y^2-x^2}{2xy};$$

therefore, in the trajectory,

$$\frac{dy}{dx} = \frac{\frac{y^2-x^2}{2xy} \pm 1}{1 \mp \frac{y^2-x^2}{2xy}} = \frac{y^2-x^2 \pm 2xy}{2xy \mp (y^2-x^2)}.$$

To integrate this, assume $x = \epsilon^\theta$, $y = z\epsilon^\theta$; then

$$z + \frac{dz}{d\theta} = \frac{z^2 \pm 2z - 1}{\mp z^2 + 2z \pm 1};$$

therefore $$\frac{d\theta}{dz} = \frac{\mp z^2 + 2z \pm 1}{\pm z^3 - z^2 \pm z - 1} = -\frac{\pm 3z^2 - 2z \pm 1}{\pm z^3 - z^2 \pm z - 1} + \frac{2}{z \mp 1};$$

therefore $$\theta = \log \frac{c(z \mp 1)^2}{\pm z^3 - z^2 \pm z - 1}, \quad c \text{ being arbitrary.}$$

Thus, $$x = \epsilon^\theta = \frac{c\left(\frac{y}{x} \mp 1\right)^2}{\pm \frac{y^3}{x^3} - \frac{y^2}{x^2} \pm \frac{y}{x} - 1},$$

that is, the equation required is

$$\pm y^3 - y^2 x \pm y x^2 - x^3 = c(y \mp x)^2,$$

or $$(\pm y - x)\left\{y^2 + x^2 - c(\pm y - x)\right\} = 0.$$

Thus the trajectory is a system of circles $x^2 + y^2 + c(x \pm y) = 0$, all passing through the fixed point and having their centres on two lines inclined at 45° to the given line. These lines are, of course, themselves included as particular cases.

II. *Solution by* Asher B. Evans, M.A.

Take the given point as origin and the given line as axis of x, and the equation to any one of the circles will be

$$y'^2 = 2ax' - x'^2, \text{ whence } a = \frac{x'^2 + y'^2}{2x'} \text{ and } \frac{dy'}{dx'} = -\frac{x'^2 - y'^2}{2x'y'}.$$

Since the tangent of the angle at which the required curve cuts each of the circles is unity, we have $\frac{dy'}{dx'} - \frac{dy}{dx} = 1 + \frac{dy'}{dx'} \cdot \frac{dy}{dx}$. Substituting the value of $\frac{dy'}{dx'}$ in this equation and omitting the accents, we have

$$(x^2 - y^2 + 2xy)\,dy + (x^2 - y^2 - 2xy)\,dx = 0.$$

Put $y = xz$, then $$\frac{dx}{x} + \frac{(1 + 2z - z^2)\,dz}{(1-z)(1+z^2)} = 0.$$

Integrating, $\log x = \log\left(\frac{z-1}{z^2+1}\right) + \log c$, therefore $x = \frac{c(z-1)}{z^2+1}$; whence $x^2 + y^2 = c(y - x)$, which is evidently the equation of the circle.

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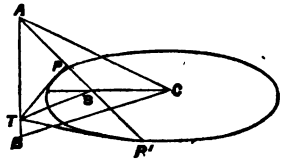
4137. (Proposed by the Rev. J. R. Wilson, M.A.)—ABC is a triangle; with C as centre, AB as directrix, and CA, CB as the directions of a pair of conjugate diameters, describe an ellipse; let S be the focus within the triangle; draw ST parallel to BC, meeting AB in T, and draw TP to touch the ellipse in P; then show that the point P lies on SA.

I. *Solution by* C. Leudesdorf, B.A.

The polar of A, being parallel to CB and passing through S (since A lies on the directrix), will be ST. Hence, T lying on the polar of A, A will lie on the polar of T, or PP′ will pass through A. And since T lies on the directrix, PP′ will also pass through S. Thus the points P, P′ lie on SA.



II. *Solution by* S. Forde; A. B. Evans, M.A.; *and others.*

Refer the ellipse to its axes; then the equations of the ellipse, and of the lines CA, CB, ST are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad y=\frac{b}{a}\tan\theta.x, \quad y=-\frac{b}{a}\cot\theta.x, \quad y=-\frac{b}{a}\tan\theta\,(x-ae).$$

In ST, put $x=\frac{a}{e}$, then $y=-b\tan\theta\left(\frac{1}{e}-e\right)$; which gives T.

Hence the polar of T is $\frac{x}{ae}-y\tan\theta\left(\frac{1-e^2}{be}\right)=1$;

and putting $x=\frac{a}{e}$, we have $y=\frac{b}{e}\cot\theta$;

therefore the polar of T passes through the intersection of CA and $x=\frac{a}{e}$, that is, through A. It will necessarily pass also through S; therefore P lies in SA.

4402. (Proposed by W. H. H. Hudson, M.A.)—If r, p, ρ be radius vector, perpendicular on the tangent, and radius of curvature of a curve, and r', p', ρ' be corresponding quantities for an inverse curve; prove that

$$\frac{r}{\rho}+\frac{r'}{\rho'}=\frac{p}{r}+\frac{p'}{r'}.$$

Solution by R. Tucker, M.A.

In my notes on *Radials*, § 37 (*Reprint*, Vol. II., p. 30), I have shown that

$$\rho'=k^2\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}\div\left\{u^2+2\left(\frac{du}{d\theta}\right)^2-u\frac{d^2u}{d\theta^2}\right\};$$

$$\therefore \quad \frac{r}{\rho}+\frac{r'}{\rho'}=\frac{u^2\left(u+\frac{d^2u}{d\theta^2}\right)}{\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}+\frac{u^3+2u\left(\frac{du}{d\theta}\right)^2-u^2\frac{d^2u}{d\theta^2}}{\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}}}=\frac{2u}{\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}^{\frac{1}{2}}};$$

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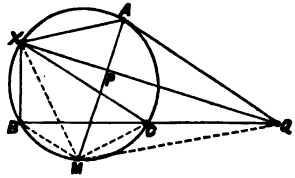
but $$\frac{p}{r}+\frac{p'}{r'}=\frac{u}{\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}^{\frac{1}{2}}}+\frac{u}{\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\}^{\frac{1}{2}}};$$

whence we get $$\frac{r}{\rho}+\frac{r'}{\rho'}=\frac{p}{r}+\frac{p'}{r'}.$$

4530. (Proposed by Professor WOLSTENHOLME, M.A.)—A triangle ABC is inscribed in a circle, and the tangent at A meets BC in Q; prove (1) that a straight line drawn through Q perpendicular to the bisector of the angle A will meet the circle in two points such that their distances from B, A, C are in geometrical progression; also (2) that the straight line through Q parallel to the bisector of the angle A will meet the circle in two points possessing the same property, provided that $a^2 > 4bc$.

I. *Solution by* Professor TOWNSEND, M.A., F.R.S.

(1). If M be either point of bisection of the arc BC, and X either point of intersection with the circle of the perpendicular QP from the point Q upon the straight line AM, it is to be shewn that $BX.CX = AX^2$; which may be readily done as follows. Drawing MB, MC, MQ, MX; since, on elementary principles, $MX^2 - AX^2 = MP^2 - AP^2 = MQ^2 - AQ^2 = MQ^2 - BQ.CQ = MB.MC = MX^2 - BX.CX$, therefore $BX.CX = AX^2$, and therefore, &c.



II. *Solution by the* PROPOSER.

(1). If x, y, z be areal coordinates of any point P on the circle, a the point where the tangent at A meets BC, and θ the angle PAB, we have

$$x : y : z = \frac{PA}{\sin A} : \frac{PB}{\sin B} : \frac{-PC}{\sin C},$$

or

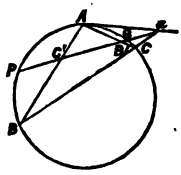
$$= \frac{PA}{\sin A} : \frac{-PB}{\sin B} : \frac{PC}{\sin C}$$

(for there must be one point between A and B, and one between A and C; in the figure P is the first, and Q the second); therefore, when $PA^2 = PB.PC$, we shall have

$$\frac{x^2}{\sin^2 A} = \frac{-yz}{\sin B \sin C},$$

but, at any point on the circle,

$$\frac{\sin^2 A}{x} + \frac{\sin^2 B}{y} + \frac{\sin^2 C}{z} = 0;$$



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therefore $$x = y\frac{\sin C}{\sin B} + z\frac{\sin B}{\sin C},$$

a straight line which, by its intersections with the circle, gives the required points. This straight line passes through

$$a, \left(x = 0, \frac{y}{\sin^2 B} + \frac{z}{\sin^2 C} = 0\right),$$

and is parallel to $$\frac{y}{\sin B} + \frac{z}{\sin C} = 0,$$

since it may be written

$$(x+y+z) - (\sin B + \sin C)\left(\frac{y}{\sin B} + \frac{z}{\sin C}\right) = 0.$$

(2). There may also be two points on the arc BC at which

$$\frac{x}{-\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C},$$

for which we shall have $$\frac{x^2}{\sin^2 A} = \frac{yz}{\sin^2 B \sin C},$$

and therefore $$x + y\frac{\sin C}{\sin B} + z\frac{\sin B}{\sin C} = 0,$$

a straight line passing through a, and equivalent to

$$(x+y+z) - (\sin B - \sin C)\left(\frac{y}{\sin B} - \frac{x}{\sin C}\right) = 0,$$

and therefore parallel to the bisector of the angle ABC. When this meets the circle, we have $$\sin^2 A = \left(\frac{\sin^2 B}{y} + \frac{\sin^2 C}{z}\right)\left(y\frac{\sin C}{\sin B} + z\frac{\sin B}{\sin C}\right),$$

or $$y^2 \sin^4 C + yz\,(2 \sin B \sin C - \sin^2 A) \sin B \sin C + z^2 \sin^4 B = 0,$$

which will give real values only when

$$(2 \sin B \sin C - \sin^2 A)^2 > 4 \sin^2 B \sin^2 C,$$

or $$\sin^2 A > 4 \sin B \sin C, \text{ or } a^2 > 4bc.$$

If the $\angle BAP = \theta$, and $CAQ = \phi$, we have

$$\frac{AP}{\sin(C-\theta)} = \frac{BP}{\sin\theta} = \frac{CP}{\sin(A+\theta)}, \text{ whence } \sin^2(C-\theta) = \sin\theta \sin(A+\theta),$$

or $$\cos 2\theta\,(\cos 2C - \cos A) + \sin 2\theta\,(\sin 2C + \sin A) = 1 - \cos A,$$

and the two roots of this equation are θ, $-(A+\phi)$, whence

$$\tan(\theta - A - \phi) = \frac{\sin 2C + \sin A}{\cos 2C - \cos A} = \cot(\tfrac{1}{2}A - C),$$

$$\theta - \phi - A = \pm\tfrac{1}{2}\pi - \tfrac{1}{2}A + C, \text{ or } \theta - \phi = \tfrac{1}{2}A + C - \tfrac{1}{2}\pi = \tfrac{1}{2}(C-B).$$

So, for P′, Q′, the two points between B and C, if $BAP' = \theta'$, $CAQ' = \phi'$, the equation is $$\sin^2(C+\theta') = \sin\theta' \sin(A-\theta'),$$

or $$\cos 2\theta'\,(\cos A + \cos 2C) + \sin 2\theta'\,(\sin A - \sin 2C) = 1 + \cos A,$$

and the roots are θ', $A - \phi'$, so that

$$\tan(\theta' - \phi' + A) = \frac{\sin A - \sin 2C}{\cos A + \cos 2C} = \tan(\tfrac{1}{2}A - C),$$

$$\theta' - \phi' + A = \tfrac{1}{2}A - C, \quad \theta' - \phi' = -\tfrac{1}{2}A - C,$$

or $$\pi - \tfrac{1}{2}A - C, \text{ } i.e., \text{ } = \tfrac{1}{2}(B-C) - \tfrac{1}{2}\pi, \text{ or } \tfrac{1}{2}(B-C) + \tfrac{1}{2}\pi.$$

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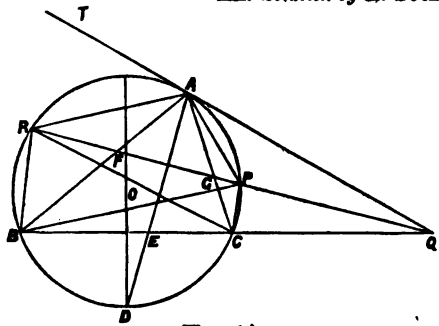
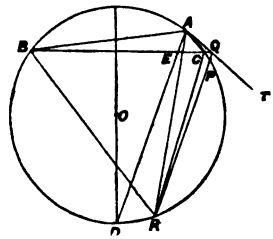
Hence it appears that PAP', QAQ' will differ by a right angle. This is a necessary consequence of PQ, P'Q' being at right angles. [See WOLSTENHOLME'S *Book of Mathematical Problems*, Question 335].

III. *Solution by* R. TUCKER, M.A.



(FIG. 1.)



(FIG. 2.)

Let the line through Q in each case meet the circle in P, R; then (Euc. VI. C.) we have

$$RB \cdot RC = 2R \text{ (perpendicular from R on BC)}$$
$$= 2R \text{ (perpendicular from R on AQ)} = RA^2,$$

since $\angle RAT = \frac{1}{2}ROA$.

In Fig. 1 the angle Q is bisected, by construction; and in Fig. 2,

$$\angle RQT = DAT = EAC + EBA = EAB + EBA = AEC = EQR,$$

and therefore QR bisects the angle BQT.

The same proof obviously holds for P in place of R.

In Fig. 2, to find when QPR cuts the circle. Let the circle be referred to BC and the diameter through D; then its equation is

$$x^2 + y^2 - 2R\cos A\, y - \tfrac{1}{4}a^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The equation of the tangent at A $(c\cos B - \frac{1}{2}a,\ c\sin B)$ is

$$xx' + yy' - R\cos A\,(y + y') - \tfrac{1}{4}a^2 = 0.$$

Making $y = 0$, we have $x = \dfrac{a(b^2+c^2)}{2(c^2-b^2)}$, and the equation to QPR is

$$y = \tan(\tfrac{1}{2}A + B)\left\{x - \frac{a(b^2+c^2)}{2(c^2-b^2)}\right\}.$$

Combining this with (1), we get

$$y^2\operatorname{cosec}^2(\tfrac{1}{2}A+B) + 2y\left\{\frac{a(b^2+c^2)}{2(c^2-b^2)}\cot(\tfrac{1}{2}A+B) - R\cos A\right\} + \frac{a^2b^2c^2}{(c^2-b^2)^2} = 0 \quad \ldots\ldots (2).$$

Now $$\frac{\sin(\frac{1}{2}A+B)}{\sin\frac{1}{2}A} = \frac{c}{BE} = \frac{b}{CE}, \quad BE = \frac{ac}{b+c};$$

whence $\sin(\frac{1}{2}A+B) = \dfrac{b+c}{a}\sin\frac{1}{2}A$, and $\cos(\frac{1}{2}A+B) = \dfrac{c-b}{a}\cos\frac{1}{2}A$;

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therefore $$\cot(\tfrac{1}{2}A + B) = \frac{c-b}{c+b}\cot\tfrac{1}{2}A.$$

Substituting in (2), we have

$$\frac{a^2}{(b+c)^2}\operatorname{cosec}^2\tfrac{1}{2}A \cdot y^2 + ay\left\{\frac{b^2+c^2}{(c+b)^2}\cot\tfrac{1}{2}A - \cot A\right\} + \frac{a^2b^2c^2}{(c^2-b^2)^2} = 0;$$

whence, for real section, we must have

$$(b^2+c^2)\cot\tfrac{1}{2}A - (c+b)^2\cot A > \operatorname{cosec}\tfrac{1}{2}A\,\frac{2abc}{c-b};$$

i.e., $(b^2+c^2)(1+\cos A) - (c+b)^2\cos A > \dfrac{4abc\cos\frac{1}{2}A}{(c-b)}$; *i.e.*, $a > \dfrac{4bc\cos\frac{1}{2}A}{c-b}$;

i.e., $$a^2(c-b)^2 > 4bc\left\{(b+c)^2 - a^2\right\};\quad \textit{i.e.},\quad a^2 > 4bc.$$

IV. *Solution by the* EDITOR.

Taking ABC as the triangle of reference for trilinear coordinates (see the figures to Mr. TUCKER's Solution), the equations of the circle, of the tangent AQ, and of the *external* and *internal* bisectors of the angle A, are respectively $$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0,\quad c\beta + b\gamma = 0,\quad \beta \pm \gamma = 0 \ldots\ldots (1, 2, 3).$$

A straight line through Q parallel to (3) must be designated by each of the two following *identical* equations:—

$$k\alpha + c\beta + b\gamma = 0,\quad l(\beta \pm \gamma) + (a\alpha + b\beta + c\gamma) = 0;$$

therefore $$\frac{a}{k} = \frac{b+l}{c} = \frac{c \pm l}{b},\ \text{whence}\ k = \mp a.$$

Thus the equations of the two lines through Q, parallel to the bisectors of the angle A, are $$\mp a\alpha + c\beta + b\gamma = 0,\ \text{or}\ \frac{b}{\beta} + \frac{c}{\gamma} \mp \frac{a\alpha}{\beta\gamma} = 0 \ldots\ldots (4),$$

where the upper sign (−) applies to the line QPR in the *first* part of the theorem (*perpendicular* to AED, Fig. 1), and the lower sign (+) to the line QPR in the *second* part (*parallel* to AED, Fig. 2). Combining (1) with the *second* form of (4), we see that at the points (P, Q) where the line QPR meets the circle, we must have

$$\frac{a}{\alpha} = \mp\frac{a\alpha}{\beta\gamma},\ \text{or (disregarding sign)}\ \frac{\beta}{\alpha} = \frac{\alpha}{\gamma};$$

therefore $$\frac{\beta\operatorname{cosec}\mathrm{PAC}}{\alpha\operatorname{cosec}\mathrm{PBC}} = \frac{\alpha\operatorname{cosec}\mathrm{PCB}}{\gamma\operatorname{cosec}\mathrm{PAB}},\ \text{or}\ \frac{\mathrm{PA}}{\mathrm{PB}} = \frac{\mathrm{PB}}{\mathrm{PC}} \ldots\ldots (5).$$

To find when the intersections of the line QPR with the circle are *real*, eliminate α from (1) and (4); then we have

$$b^2\gamma^2 + (2bc \pm a^2)\beta\gamma + c^2\beta^2 = 0 \ldots\ldots (6);$$

and the roots of (6) will be *real*, provided that

$$(2bc \pm a^2)^2 > 4b^2c^2,\ \text{or}\ a^2 \pm 4bc > 0 \ldots\ldots (7).$$

As the upper signs throughout apply to the first case, we see from (7) that the intersections here are *always* real; but from (7) we likewise see that the intersections in the second case are real only so long as $a^2 > 4bc$.

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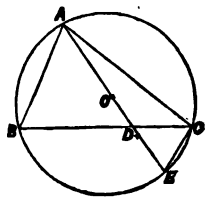
4500. (Proposed by J. J. WALKER, M.A.)—Prove that the intercept on the diameter of the circle circumscribing the plane triangle ABC between the angle A and the opposite side BC is equal to

$$\frac{a\cos A + b\cos B + c\cos C}{\sin 2B + \sin 2C}.$$

Solution by A. J. P. SHEPHERD; R. F. DAVIS; *and others.*



Let D be the point of intersection of the diameter AOE and the base BC; then

$\angle OAC = \frac{1}{2}\pi - B$, therefore $ADB = C + \frac{1}{2}\pi - B$,

therefore $$AD = \frac{c\sin B}{\cos(B-C)}$$

$$= \frac{2c\sin B\sin A}{\sin 2B + \sin 2C} = \frac{2c}{\sin C}\cdot\frac{\sin A\sin B\sin C}{\sin 2B + \sin 2C}$$

$$= \frac{c}{2\sin C}\cdot\frac{\sin 2A + \sin 2B + \sin 2C}{\sin 2B + \sin 2C}$$

$$= \frac{a\cos A + b\cos B + c\cos C}{\sin 2B + \sin 2C}.$$

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